APPLICATIONS OF A VIETORIS-BEGLE THEOREM FOR MULTI-VALUED MAPS TO THE COHOMOLOGY OF HYPERSPACES

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Let C(X) denote the hyperspace of subcontinua of the continuum X. J. L. Kelley [1] observed that H. Whitney had defined a monotone map μ: C(X) $\to$ R satisfying the three conditions

(1) $\mu(\{x\}) = 0$ for each point x in X,

(2) $\mu(X) = 1$, and

(3) $\mu(A) < \mu(B)$ whenever $A \subset B$ and $A \neq B$.

A function μ satisfying these conditions is called a *Whitney map.* (It would be more accurate to call it a Whitney-Kelley map.) The collection $\{\mu^{-1}(t)\colon 0 \leq t \leq 1\}$ is called the set of *Whitney subcontinua* of C(X) or the set of *Whitney continua associated with* X. Note that $\mu^{-1}(1)$ is the singleton set $\{X\}$ and that $\mu^{-1}(0)$ is the set of degenerate subcontinua of X. Since the map of X into C(X) defined by sending a point x to the degenerate continuum $\{x\}$ is an isometry of X into C(X), it follows that $\mu^{-1}(0) \cong X$.

The following general problem naturally presents itself: Suppose that X has a topological property P. What can one say about $\mu^{-1}(t)$? In particular, does $\mu^{-1}(t)$ have P?

The following example shows that X and $\mu^{-1}(t)$ need not be homeomorphic; in fact, there are numbers s and t such that X cannot be mapped onto $\mu^{-1}(t)$ and $\mu^{-1}(s)$ cannot be mapped onto X.

Example. Let X be the planar continuum obtained from the standard topologist's $\{\sin 1/x\}$-curve by identifying the points (0, 1) and (0, -1). The continuum X is pictured in Figure 1. If ε is a small positive number, then $\mu^{-1}(1 - \varepsilon) \cong [0, 1]$, while $\mu^{-1}(\varepsilon)$ is homeomorphic to the planar continuum pictured in Figure 2. To see this last fact, note that if a segment of the circle has (0, -1) as an interior point, then (for sufficiently small ε) there is no family of subcontinua of the $\{\sin 1/x\}$-curve that converges to it. Therefore X cannot be mapped onto $\mu^{-1}(\varepsilon)$, while $\mu^{-1}(1 - \varepsilon)$ cannot be mapped onto X.

Our example shows that X and $\mu^{-1}(t)$ need not be cohomologically equivalent. There is, however, a relationship between the first cohomology groups of X and $\mu^{-1}(t)$ that can be stated roughly as follows: As we go higher into the hyperspace, no new one-dimensional holes are created, and perhaps some one-dimensional holes are swallowed. This vague conjecture finds formulation as the following theorem:

THEOREM. *For each continuum* X *and each* t *in* [0, 1], *there is an induced injection*

$$\gamma^*\colon H^1(\mu^{-1}(t)) \to H^1(X).$$

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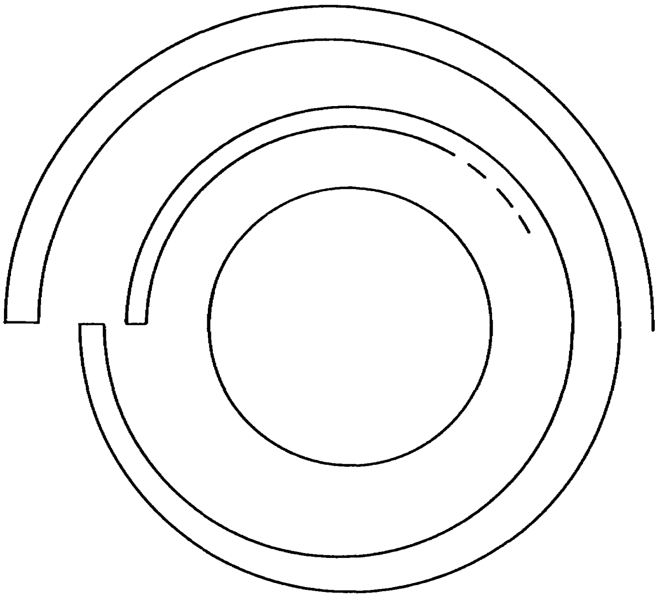

Figure 1.

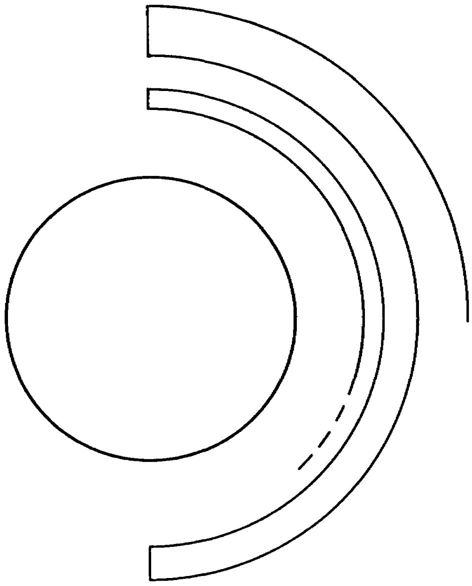

Figure 2.

A continuum X is a compact, connected, nonvoid metric space. 2^X is the set of all nonempty, closed subsets of X with the topology induced by the Hausdorff metric. The subspace C(X) of 2^X consists of all the subcontinua of X.

$H^n(X)$ denotes the reduced nth Alexander-Čech cohomology group of the continuum X. A continuum X is *acyclic* if $H^n(X) = 0$, for $n \geq 0$. A set-valued function F from the continuum X to the continuum Y is a function from X into 2^Y. It is said to be *upper-semicontinuous* if $\{x: F(x) \subset U\}$ is open for each open set U in Y.

See [5] or [2] for information about the carry-over of other topological properties from X to $\mu^{-1}(t)$.

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1. APPLICATIONS OF THE COHOMOLOGY OF STRUCTURES TO ACYCLICITY IN HYPERSPACES

For each point Z of C(X), we define C(X; Z) to be the set of all subcontinua Y of X such that $Z \subset Y$. Then the continuum C(X; Z) is a topological semilattice with identity Z, and therefore C(X; Z) is contractible.

We shall now review Lawson's definition [3] of a cohomology theory for a structure and investigate the applications of this theory to C(X).

A nonempty collection Σ of closed subsets of a continuum is called a *structure* if Σ is closed with respect to finite unions, finite intersections, and intersections of towers ordered by inclusion.

We find two interesting structures in C(X). If $\mathscr{A}$ is a closed subset of C(X), define $M(\mathscr{A}) = \cup \{C(X; Z): Z \in \mathscr{A}\}$. Let

$$\Sigma_1 = \{M(\mathscr{A}): \mathscr{A} \text{ is a closed subset of } C(X)\}.$$

Each set $M(\mathscr{A})$ is closed in C(X) [4]. Furthermore, M is, in some sense, a closure operator, in that

$$M(A \cup B) = M(A) \cup M(B)$$

and

$$M\left(\bigcap M(A_\alpha)\right) = \bigcap M(A_\alpha),$$

where α runs through some index set. Hence Σ_1 is a structure.

The second structure occurs in C(X; Z), where Z is an arbitrary point of X. If A is a point of C(X; Z), define

$$L(A) = \{Y \in C(X; Z) : Y \subset A\}.$$

If $\mathscr{A}$ is a closed subset of C(X; Z), define

$$L(\mathscr{A}) = \bigcup \{L(A) : A \in \mathscr{A}\}.$$

Finally, define

$$\Sigma_2 = \{L(\mathscr{A}) : \mathscr{A} \text{ is a closed subset of } C(X; Z)\}.$$

Each $L(\mathscr{A})$ is closed, and the family Σ_2 is closed under finite unions and arbitrary intersections. Therefore Σ_2 is a structure.

If Σ is a structure on a continuum, then a closed set $P \in \Sigma$ is called an *indecomposable set* if $P = A \cup B$ for some A and B in Σ implies $P = A$ or $P = B$. The indecomposable sets of Σ_1 are the sets of the form M(Z), where $Z \in C(X)$. The indecomposable sets of Σ_2 are the sets of the form L(A), where $A \in C(X; Z)$. The indecomposable sets of both structures are acyclic. We now state a version of Theorem 8.1 of [3].

THEOREM 1. *Let* Σ *be a structure on a topological space, and let* H *and* $\overline{H}$ *be continuous cohomologies on* Σ. *If* τ *is a homomorphism from* H *to* $\overline{H}$ *that is an isomorphism for all indecomposable sets of* Σ, *then* τ *is an isomorphism for every* $S \in \Sigma$.

Theorem 1 is applicable to hyperspaces in the following manner.

THEOREM 2. *If* Σ *is one of the structures* Σ_1 *and* Σ_2 *defined above, then each member* S *of* Σ *is acyclic.*

Proof. Define H on Σ to be reduced Alexander cohomology, and $\overline{H}$ on Σ to be trivial cohomology. Both trivial cohomology and reduced Alexander cohomology are continuous cohomologies on Σ. Consider the natural homomorphism τ from H to $\overline{H}$. Each indecomposable set of Σ is acyclic, so that τ is an isomorphism on indecomposable sets. Therefore τ is an isomorphism for each $S \in \Sigma$; that is, each $S \in \Sigma$ is acyclic. This completes the proof.

2. THE COHOMOLOGY GROUPS OF WHITNEY CONTINUA

Consider the following extension of the Vietoris-Begle theorem to set-valued maps.

THEOREM 3. *Let* n *be a nonnegative integer, let* X *and* Y *be compact Hausdorff spaces, and let* $F: X \to Y$ *be an upper-semicontinuous, set-valued surjection that satisfies the two conditions*

(1) $H^k(F(x)) \cong 0$ *for all* x *in* X *and for all integers* k *such that* $0 \leq k \leq n+1$,

(2) $H^k(F^{-1}(y)) \cong 0$ *for all* y *in* Y *and for all integers* k *such that* $0 \leq k \leq n$.

Then there is a morphism $F^*: H^*(Y) \to H^*(X)$ *in dimensions* 0 *through* $n+1$ *such that*

(3) $F^*: H^k(Y) \to H^k(X)$ *is an isomorphism for* $0 \leq k \leq n$, *and*

(4) $F^*: H^{n+1}(Y) \to H^{n+1}(X)$ *is a monomorphism.*

Proof. Let $G = \{(x, y): y \in F(x)\}$ be the graph of F. The set G is closed in $X \times Y$, because F is upper-semicontinuous. Let $p: G \to X$ and $q: G \to Y$ be the projection maps, and let $0 \leq k \leq n+1$. Since $H^k(F(x)) \cong 0$, it follows from the Vietoris-Begle theorem that $p^*: H^k(Y) \to H^k(X)$ is an isomorphism. Define $F^*: H^k(Y) \to H^k(X)$ by $F^* = (p^*)^{-1} \circ q^*$. Since $H^k(F^{-1}(y)) \cong 0$ for $0 \leq k \leq n$, it follows from the Vietoris-Begle theorem that q^* is an isomorphism in dimensions 0 through n and a monomorphism in dimension $n+1$. Thus the same is true of F^*.

For each point Z of C(X), let $C^t_Z = \{A \in \mu^{-1}(t): Z \subset A\}$. Note that $C^t_Z = M(Z) \cap \mu^{-1}(t)$. In [5], we showed that if p is a degenerate subcontinuum of X, then C^t_p is an arcwise connected continuum. The same proof shows that C^t_Z is an arcwise connected continuum, provided that $t \geq \mu(Z)$ (if $t < \mu(Z)$, then $C^t_Z = \emptyset$). Moreover, we can obtain the following additional information on these continua.

THEOREM 4. *For each point* Z *in* C(X) *and for each* t *in* [0, 1] *satisfying* $t \geq \mu(Z)$, *the continuum* C^t_Z *is acyclic.*

Proof. Consider the pair $\{M(C^t_Z), L(C^t_Z)\}$ of subsets of C(X; Z). For an integer $n \geq 0$, consider the relevant part of the reduced Mayer-Vietoris sequence

$$H^n(M(C^t_Z)) \oplus H^n(L(C^t_Z)) \to H^n(C^t_Z) \to H^{n+1}(C(X; Z))$$

for this pair. $H^n(M(C^t_Z)) = H^n(L(C^t_Z)) = 0$, by Theorem 2, and $H^{n+1}(C(X; Z)) = 0$. Thus $H^n(C^t_Z) = 0$. Thus C^t_Z is acyclic, and the theorem is proved.

Consider the set-valued function $\gamma^t_s: \mu^{-1}(s) \to \mu^{-1}(t)$ $(s \leq t)$, from the Whitney continuum $\mu^{-1}(s)$ to the Whitney continuum $\mu^{-1}(t)$, defined by $\gamma^t_s(Z) = C^t_Z$. We showed in [5, Theorem 4.3] that the map γ^t_0 is an upper-semicontinuous, continuum-valued function; the proof there is valid for other values of s as well.

THEOREM 5. *For each* s *and* t *in* [0, 1] $(s \leq t)$, *the set-valued map* $\gamma^t_s: \mu^{-1}(s) \to \mu^{-1}(t)$ *induces a monomorphism* $(\gamma^t_s)^*: H^1(\mu^{-1}(t)) \to H^1(\mu^{-1}(s))$. *If* $\mu^{-1}(s)$ *is a curve* (*that is, a one-dimensional continuum*), *if* $t \neq 1$, *and if* $H^1(Y) = 0$ *for each proper subcontinuum* Y *of* $\mu^{-1}(s)$, *then* $(\gamma^t_s)^*: H^*(\mu^{-1}(t)) \to H^*(\mu^{-1}(s))$ *is an isomorphism.*

Proof. Each $\gamma^t_s(Z)$ is acyclic, by Theorem 4. If $t \neq 1$ and if B is a point of $\mu^{-1}(t)$, then $(\gamma^t_s)^{-1}(B)$ is the proper subcontinuum $C(B) \cap \mu^{-1}(s)$ of $\mu^{-1}(s)$. The theorem now follows from Theorem 3.

COROLLARY 6. *The map* γ^t_0 *induces a monomorphism*

$$(\gamma^t_0)^*: H^1(\mu^{-1}(t)) \to H^1(X).$$

COROLLARY 7. *If* X *is an acyclic curve, then* $\mu^{-1}(t)$ *is acyclic* (*though not necessarily a curve*).

The next corollary is proved by different methods in [5].

COROLLARY 8. *If* X *is a circle-like continuum and* $t \neq 1$, *then* X *and* $\mu^{-1}(t)$ *are cohomologically equivalent.*

Proof. Each proper subcontinuum of a circle-like continuum is arc-like and hence acyclic.

Question. Is $(\gamma_s^t)^*: H^n(\mu^{-1}(t)) \to H^n(\mu^{-1}(s))$ a monomorphism, for $n > 1$?

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