SEMILOCAL GROUP RINGS AND TENSOR PRODUCTS

John Lawrence

1. INTRODUCTION

Two central problems concerning semilocal group rings have been attacked in recent years. First, if $F[G]$ is semilocal, is G locally finite? Second, if $F[G]$ is semilocal, is G a finite extension of a p-group, where char $F = p$? Virtually no progress has been made on the first question, although S. M. Woods has proved that G must be a torsion group [12]. The first question is by far the deeper of the two. In fact, J. M. Goursaud and D. S. Passman have independently proved that an affirmative answer to the first question implies an affirmative answer to the second question. More explicitly, they proved that if $F[G]$ is semilocal and G is locally finite, then G is a finite extension of a p-group [3], [7]. However, the second question has been answered in the affirmative in cases where an answer to the first question is unknown. J. Valette proved that if F is an uncountable, algebraically closed field and $F[G]$ is semilocal, then G is a finite extension of a p-group [11].

Further results on local and semilocal group rings appear in [1] and [8].

Another problem that has been examined recently is the question when the tensor product of algebras is local. In [10], necessary and sufficient conditions were given for the tensor product of two commutative algebras to be local. Among other things, M. E. Sweedler showed that if a tensor product $A \otimes_F B$ of two commutative algebras over a field is local, then one of the algebras must be algebraic over F. Several of the results of Sweedler's paper were generalized in [6], where it was proved that in certain cases "local finiteness" is necessary (and not simply "algebraic").

In this paper, we prove that if the tensor product of two algebras (not necessarily commutative) over a field is semilocal, then at least one of the two algebras is algebraic over F. We then use this result to answer affirmatively the second question on semilocal group rings, in several new cases.

2. NOTATION, CONVENTIONS, AND BACKGROUND

All rings are associative with unity $1 \neq 0$. The Jacobson radical of a ring R is denoted by $J(R)$. A ring is said to be *semilocal* if $\overline{R} = R/J(R)$ is Artinian. R is *local* if $\overline{R} = R/J(R)$ is a division ring. In what follows, F will denote a field and G a group; $F[G]$ will denote the group ring of the group G over the field F.

Recall that a *valuation* v on a ring R is a map $v: R \to \mathbb{R}^+$ ($\mathbb{R}^+$ denotes the space of nonnegative reals) satisfying the conditions

1. $v(1) = v(-1) = 1$, $v(x) = 0$ if and only if $x = 0$,
2. $v(a + b) \leq \max[v(a), v(b)]$, for all $a, b \in R$,

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3. $v(ab) = v(a)v(b)$, for all $a, b \in R$.

If x is a prime in the unique-factorization domain R, then every nonzero element $a \in R$ can be written uniquely in the form $a = x^n y$, where x does not divide y. Define $v(a) = 2^{-n}$. This is the x-*adic valuation.*

If R is a ring with valuation v, and M is a left R-module, then a *valuation* u on M is a map $u: M \to \mathbb{R}^+$ satisfying the conditions

1. $u(a + b) \leq \max[u(a), u(b)]$, for all $a, b \in M$,
2. $u(ra) = v(r)u(a)$, for all $r \in R$, $a \in M$.

The following theorem is useful for defining valuations on modules.

THEOREM 1 [5]. *Suppose that* R *is a left Ore domain,* M *is a left* R-*module and* N *is a submodule of* M. *Then every valuation on* N *can be extended to a valuation on* M.

Assume that A is an F-algebra with subalgebra R, and that v is a valuation on the module ${}_RA$. Let B be any other F-algebra with F-basis $\{b_i\}$. Then we can extend v to a valuation u on ${}_R(A \otimes_F B)$ by defining

$$u\left(\sum (a_i \otimes b_i)\right) = \max\{v(a_i)\}.$$

This is called the *max-extension* of v on ${}_R(A \otimes_F B)$. If $\alpha = \sum (a_i \otimes b_i) \in A \otimes_F B$, then $\operatorname{supp} \alpha = \{b_i \mid a_i \neq 0\}$, and a_i is the *coefficient* of b_i.

THEOREM 2 [5]. *Let* A *be an* F-*algebra with subalgebra* R, *let* v *be a valuation on* ${}_RA$ *such that* $v(f) = 1$ *for all nonzero* $f \in F$, *let* B *be an* F-*algebra with basis* $\{b_i\}$, *and let* u *be the max-extension of* v *on* ${}_R(A \otimes_F B)$. *Suppose* $\alpha \in A \otimes_F B$ *satisfies the conditions*

1. *all coefficients of* α *are in* R,
2. $1 - \alpha$ *is right-invertible with inverse* β,
3. $u(\alpha) = s < 1$.

Let $\delta(n) = 1 + \alpha + \cdots + \alpha^n$. *If there exists* n *such that*

$$v(\textit{coefficient of } b_i \textit{ in } \delta(n)) > v(\textit{coefficient of } b_i \textit{ in } \alpha^m),$$

for all $m > n$, *then* $b_i \in \operatorname{supp} \beta$.

A lemma of Woods gives an interesting property of semilocal rings.

LEMMA 3 [12]. *Let* R *be a semilocal ring. Given* $x \in R$, *define a sequence* $\{x_n\}$ *inductively by*

$$x_1 = x, \qquad x_{n+1} = x_n - x_n^2.$$

Then, for some n, *the element* $1 - x_n$ *is right-invertible* (*hence invertible*) *in* R.

3. MAIN THEOREMS

In this section, we prove our main theorems. The first gives a rather general class of elements that are not invertible in tensor products. We then use Woods's Lemma to prove the main theorem stated in the introduction.

THEOREM 4. *Suppose* A *and* B *are nonalgebraic* F-*algebras. Let* $x \in A$ *and* $y \in B$ *be transcendental elements. Then the element*

$$1 - f_1(x \otimes y) - f_2(x \otimes y)^2 - \cdots - f_k(x \otimes y)^k \qquad (f_i \in F)$$

is right-invertible in $A \otimes_F B$ *if and only if* $f_1 = f_2 = \cdots = f_k = 0$.

Proof. Let $\alpha = \sum_{i=1}^{k} f_i(x \otimes y)^i$. Extend the set $\{1, y, y^2, \cdots\}$ to an F-basis S of B. Since $x \in A$ is transcendental, we have an x-adic valuation v on $R = F[x]$. Extend this to ${}_RA$ (Theorem 1), and let u be the max-extension of v on ${}_R(A \otimes B)$, using the basis S. For all $n \in N$, let $\delta(n) = 1 + \alpha + \cdots + \alpha^n$. Let β be the right inverse of $1 - \alpha$. Since $\delta(n)$ is a polynomial in $(x \otimes y)$, we have the formula

$$\delta(n) = \sum_{i=0}^{kn} f_i^{(n)}(x^i \otimes y^i).$$

For all sufficiently large n, say $n \geq m$, $y^n \notin \operatorname{supp} \beta$. Let

$$\pi = \sum_{i=0}^{m} f_i^{(m)}(x^i \otimes y^i),$$

so that π is an initial segment of $\delta(m)$. Let $\delta(n) = \pi + \sigma(n)$, for $n \geq m$. Since $\delta(n) - \delta(m)$ is a polynomial in $(x \otimes y)$ divisible by $(x \otimes y)^{m+1}$, and since $\delta(m) - \pi$ is also divisible by $(x \otimes y)^{m+1}$, we see that $\sigma(n)$ is a polynomial in $(x \otimes y)$ divisible by $(x \otimes y)^{m+1}$.

We now claim that for all $n > m$, $\sigma(n)$ is divisible by $(x \otimes y)^n$. Observe that for each $t > n$, α^t is divisible by $(x \otimes y)^t$, so that it has no y^i in its support, for $i \leq n$. Thus, if y^i is in the support of $\sigma(n)$ for $m < i \leq n$, then y^i is in the support of $\delta(n)$ (by the disjointness of the supports of π and $\sigma(n)$). This yields the inequality

$$v(\text{coefficient of } y^i \text{ in } \delta(n)) > v(\text{coefficient of } y^i \text{ in } \alpha^t)$$

for all $t > n$, so that Theorem 2 yields the relation $y^i \in \operatorname{supp} \beta$, a contradiction.

We now restrict our attention to $F[x] \otimes F[y]$. There is a natural embedding of this ring into the power series ring $F[[x, y]]$. In the power series ring, $1 - \alpha$ is invertible with inverse $\beta' = 1 + \alpha + \cdots$. By our previous result, $\beta' = \pi$, which implies that $1 - \alpha$ is invertible in $F[x] \otimes F[y]$. It is clear that $\alpha = 0$, our desired result.

COROLLARY 5 [5]. *Suppose* A *and* B *are* F-*algebras and* A *is not algebraic. Then* $J(A \otimes_F B) \cap B$ *is a nil ideal of* B. (*Here* B *is identified with* $1 \otimes_F B$.)

Proof. Suppose the two elements $a \in A$ and $b \in J(A \otimes B) \cap B$ are transcendental. Then $1 - (a \otimes b)$ is right-invertible, a contradiction. Since A is not algebraic, we may assume that b is algebraic. Since b is in the Jacobson radical, it is nilpotent.

THEOREM 6. *Suppose* A *and* B *are* F-*algebras. If* $A \otimes_F B$ *is semilocal, then either* A *is* F-*algebraic or* B *is* F-*algebraic.*

Proof. If not, choose transcendental elements $x \in A$ and $y \in B$. Let $\alpha = (x \otimes y)$. Now use Lemma 3 and Theorem 4 to reach a contradiction.

4. COROLLARIES ON GROUP RINGS

PROPOSITION 7. *Suppose* G *is a group and* F *is a field transcendental over a subfield* K. *If the group ring* F[G] *is semilocal, then* K[G] *is algebraic.*

THEOREM 8 [9]. *Let* A *and* B *be* F-*algebras. If* $A \otimes_F B$ *is semilocal, then* A *is semilocal.*

Note. A. Rosenberg and D. Zelinsky deal with algebras not necessarily containing a unity. Their theorem states that if $A \otimes_F B$ is semilocal, then either $J(B) = B$ or A is semilocal.

THEOREM 9. *Suppose* G *is a group and* F *is a field transcendental over the algebraic closure of its prime subfield* K. *If the group ring* F[G] *is semilocal, then* G *is a finite extension of a* p-*group* (char F = p).

Proof. By Proposition 7 and Theorem 8, L[G] is semilocal and algebraic, where L is the algebraic closure of K. Thus $L[G]/J(L[G])$ is a finite direct sum of matrix rings over L. Let $^{-}: G \to L[G]/J(L[G])$ be the canonical map. Then $\ker(-) = \{g \in G: 1 - g \in J(L[G])\}$. Thus $G/\ker(-)$ is locally finite [2, p. 252]; hence, $G/\ker(-)$ is a finite extension of a p-group [3]. But $\ker(-)$ is a p-group; hence, G is a finite extension of a p-group.

PROPOSITION 10. *Let* G *and* H *be groups, and let* F *be an algebraically closed field. If* $F[G \times H]$ *is semilocal, then either* G *is a finite extension of a* p-*group, or* H *is a finite extension of a* p-*group* (char F = p).

Proof. F[G] and F[H] are homomorphic images of $F[G \times H]$, hence, semilocal. The rest of the proof is similar to the proof of Theorem 9.

We can strengthen Proposition 10 slightly when the characteristic of F is zero. I. N. Herstein [4] has noted that if F[G] is algebraic and char F = 0, then G is locally finite. Therefore, if char F = 0 and $F[G \times H]$ is semilocal, then either G is a finite group or H is a finite group.

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Carleton University
Ottawa, Ontario, Canada
and
The University of Chicago
Chicago, Illinois 60637