BASES IN COMPLETELY DISTRIBUTIVE LATTICE-ORDERED GROUPS

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1. INTRODUCTION

An element s of a lattice-ordered group (ℓ-group) G is *basic* (see [4]) if $s > 0$ and the closed interval $[0, s]$ is totally ordered. An ℓ-group G *has a basis* if every element $g > 0$ exceeds some basic element (every maximal disjoint set of basic elements is then a *basis*). An ℓ-group G is *completely distributive* (see [2], [3], [11], [12]) if the relation

$$\bigwedge\left\{ \bigvee \{g_{ij} \mid j \in J\} \mid i \in I\right\} = \bigvee\left\{ \bigwedge\{g_{i(if)} \mid i \in I\} \mid f \in J^I\right\}$$

holds whenever $\{g_{ij} \mid i \in I,\ j \in J\} \subseteq G$ is such that all the indicated joins and meets exist. By [8, p. 5.18, Theorem 5.8], every ℓ-group that has a basis is completely distributive. We wish to investigate the converse of this result for representable ℓ-groups. To this end, we define the bi-prime group of an ℓ-group (Section 2), which we characterise as the unique convex ℓ-subgroup of G that is maximal with respect to being generated by its intersections with any two minimal prime subgroups. For each ℓ-group, the bi-prime group contains all the basic elements (Section 2), but it may be the whole group, even if the group is nontrivial and abelian and contains no basic elements. For a completely distributive, representable ℓ-group, however, the latter situation cannot happen: that is, the bi-prime group of such a group has a basis (Theorem 3.1). In Section 4, we consider some examples, including one of an ℓ-group that is abelian and completely distributive and contains no basic elements.

Notation and Terminology. For terminology and notation left undefined, see G. Birkhoff [1], L. Fuchs [10], or P. Conrad [8].

A poset in which every pair of elements is comparable is a *totally ordered* set; a totally ordered group is an *o-group*. We write functions on the right, and denote the empty set by □. We use N, Z, and R to denote, respectively, the natural numbers, the integers, and the real numbers. An L-*isomorphism* of ℓ-groups, $\ell\colon G \to H$, is a one-to-one function such that both it and its inverse $\ell^{-1}\colon G\ell \to H$ preserve the group operations and arbitrary meets and joins.

We denote the cardinal product of a collection $\{T_\gamma \mid \gamma \in \Gamma\}$ of ℓ-groups by $|\Pi|\{T_\gamma \mid \gamma \in \Gamma\}$; if Γ is finite, say $\Gamma = \{1, 2\}$, then we write $T_1 \,|\times|\, T_2$. The *lexicographic product* $L \overleftarrow{\times} T$ of an ℓ-group L by an o-group T is the group $L \times T$ ordered by $(x, y) \leq (a, b)$ if $y < b$ or $y = b$ and $x \leq a$. Under this ordering, $L \overleftarrow{\times} T$ is an ℓ-group.

Now let G be an ℓ-group. If $A, B \subseteq G$, then we denote the convex ℓ-subgroup generated by A by $\langle A \rangle$; $\langle A, B \rangle \equiv \langle A \cup B \rangle$. For $A \subseteq G$, we write

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$$A^+ \equiv \{a \in A \mid a \geq 0\},$$

$$A^\perp \equiv \{x \in G \mid |x| \wedge |a| = 0 \text{ for all } G \in A\}.$$

Then $A^\perp$ is a convex ℓ-subgroup of G (see [7]). If $A = \{a\}$, we let $A^\perp = a^\perp$. If $M \subseteq G$ is a convex ℓ-subgroup of G, then L(M), the set of left cosets of M, forms a lattice under the operations

$$(a + M) \vee (b + M) = (a \vee b) + M, \qquad (a + M) \wedge (b + M) = (a \wedge b) + M.$$

The convex ℓ-subgroup M of G is *prime* if and only if L(M) is totally ordered; if M is normal, then M is prime if and only if G/M is an o-group. Let $\{P_\phi \mid \phi \in \Phi(G)\}$ be the set of minimal prime subgroups of G. If each P_ϕ is normal, then G is said to be *representable*. A subset $X \subseteq G$ is *closed* if, whenever $\{x_\alpha\} \subseteq X$ is such that $\bigvee_G x_\alpha \in G$, then $\bigvee_G x_\alpha = \bigvee_X x_\alpha \in X$, and similarly for meets. For each $\phi \in \Phi(G)$, let

$$P_\phi^* = \left\langle \left\{ g \in G \mid g = \bigvee_G g_\alpha \text{ for some subset } \{g_\alpha\} \subseteq P_\phi^+ \right\} \right\rangle.$$

Then by [3, Lemma 3.2], for each $\phi \in \Phi(G)$, P_ϕ^* is a closed, convex ℓ-subgroup of G that is normal if P_ϕ is normal. An important result of R. D. Byrd and J. T. Lloyd [3, Corollary 2.8] states that G is completely distributive if and only if $\bigcap \{P_\phi^* \mid \phi \in \Phi(G)\} = \{0\}$. Thus, if G is completely distributive and representable, the function $\rho: G \to \left|\prod\right| \{G/P_\phi^* \mid \phi \in \Phi(G)\}$, defined by $(\phi)(g\rho) = g + P_\phi^*$, is an L-isomorphism of G into the cardinal product of the o-groups $\{G/P_\phi^* \mid \phi \in \Phi(G)\}$ (see [8, p. 5.16, Corollary].

Convention. Recall that if $\{P_\alpha \mid \alpha \in A\}$ is an indexed set of subsets of a set X, and if $A = \square$, then $\bigcap \{P_\alpha \mid \alpha \in A\} = X$.

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2. THE BI-PRIME GROUP OF AN ℓ-GROUP

In this section we define a convex ℓ-subgroup of an ℓ-group G in terms of the lattice of prime subgroups of G. This ℓ-subgroup may be characterised as the unique convex ℓ-subgroup that is maximal with respect to being generated by its intersections with any two minimal prime subgroups, and it must contain all the basic elements of G.

Let G be an ℓ-group. As in Section 1, let $\{P_\phi \mid \phi \in \Phi(G)\}$ be the set of minimal prime subgroups of G. We let

$$B(G) = \bigcap \{\langle P_\phi, P_\omega \rangle \mid \phi, \omega \in \Phi(G),\ \phi \neq \omega\}$$

be the *bi-prime group* of G. Although, as is well known, $\bigcap P_\phi = \{0\}$, B(G) may be quite large: for instance, $B(R \,|\times|\, R) = R \,|\times|\, R$, and $B((R \,|\times|\, R) \overleftarrow{\times} Z) = R \times R \times \{0\}$.

PROPOSITION 2.1. B(G) *is unique among maximal convex* ℓ*-subgroups* H *of* G *that satisfy the condition*

(*) *for all* $\alpha, \beta \in \Phi(G)$ *such that* $\alpha \neq \beta$,

$$H = \langle H \cap P_\alpha, H \cap P_\beta \rangle.$$

Proof. It is easy to see that if H is a convex ℓ-subgroup of G satisfying (*), then $H \subseteq B(G)$. Thus, it suffices to show that B(G) satisfies (*). By [8, p. 1.6, Theorem 1.4], the set of convex ℓ-subgroups of G is a distributive lattice when ordered by inclusion. Thus, if $\alpha \neq \beta$ in $\Phi(G)$,

$$\langle B(G) \cap P_\alpha, B(G) \cap P_\beta \rangle = B(G) \cap \langle P_\alpha, P_\beta \rangle.$$

But by definition of B(G),

$$B(G) \cap \langle P_\alpha, P_\beta \rangle = B(G),$$

and hence B(G) satisfies (*).

COROLLARY 2.2. *Let* G *be an* ℓ*-group. Let* $\alpha, \beta \in \Phi(G)$ *be such that* $\alpha \neq \beta$. *Then*

$$B(G) = \langle B(G) \cap P_\alpha^*, B(G) \cap P_\beta^* \rangle.$$

Proof. Clearly, $B(G) \supseteq \langle B(G) \cap P_\alpha^*, B(G) \cap P_\beta^* \rangle$. Since $P_\alpha \subseteq P_\alpha^*$ and $P_\beta \subseteq P_\beta^*$, by Proposition 2.1,

$$B(G) = \langle B(G) \cap P_\alpha, B(G) \cap P_\beta \rangle \subseteq \langle B(G) \cap P_\alpha^*, B(G) \cap P_\beta^* \rangle.$$

PROPOSITION 2.3. *If* s *is a basic element of an* ℓ*-group* G, *then* $s \in B(G)$.

Proof. Let s be a basic element of G. By [8, p. 3.13, Theorem 3.1], $s^\perp$ is a minimal prime subgroup of G. Thus, there exists $\omega \in \Phi(G)$ such that $s^\perp = P_\omega$. If G is totally ordered, $\Phi(G)$ has only one element, and hence $s \in G = B(G)$, by definition of B(G). Otherwise, suppose that $\phi \in \Phi(G) \setminus \{\omega\}$ is such that $s \notin P_\phi$. If $t \wedge s = 0$, then $t \in P_\phi$, since P_ϕ is prime. Thus $P_\omega = s^\perp \subseteq P_\phi$. Since P_ϕ is minimal and P_ω is prime, this implies that $P_\phi = P_\omega$. This contradicts our choice of ϕ, and thus $s \in P_\phi$. Therefore, for all $\alpha, \beta \in \Phi(G)$ with $\alpha \neq \beta$, $s \in \langle P_\alpha, P_\beta \rangle$; that is, $s \in B(G)$.

Example 2.4. We construct an ℓ-group E such that B(E) = E contains no basic elements.

Let $L = |\Pi| \{Z \mid n \in N\}$, and let

$$E = \{f \in L \mid \text{there exists } k \in N \text{ such that } (k+n)f = (n)f \text{ for all } n \in N\}.$$

Clearly, E is an ℓ-subgroup of L, and it is easy to see [9, p. 165, Example 1] that E contains no basic elements. Furthermore, since each $g \in E$ is periodic, there exists $m_g \in N$ such that $(n)g \leq m_g$ for all $n \in N$. Thus, if $f, g \in G^+ \setminus \{0\}$, then $(n)g \leq m_g(n)f$ for all $n \in N$ such that $(n)f \neq 0$. By [8, p. 2.17, Theorem 2.4], this implies that each proper prime subgroup is both maximal and minimal. Therefore, $B(E) = E$.

3. COMPLETELY DISTRIBUTIVE, REPRESENTABLE ℓ-GROUPS

This section is devoted to showing that the pathology of Example 2.4 disappears in completely distributive, representable ℓ-groups. Specifically, we shall prove the following result.

THEOREM 3.1. *If* G *is a completely distributive, representable* ℓ*-group, then* $B(G)$ *has a basis.*

If G is an o-group, then G is representable and completely distributive, and $B(G) = G$ has a basis. Thus, we may assume for the proof of Theorem 3.1 that G is not an o-group. In particular, this means that $\Phi(G)$ has at least two distinct elements.

The proof of Theorem 3.1 then relies on the notation described in Section 1. Thus, recall that if $F = \left|\prod\right| \{G/P_\phi^* \mid \phi \in \Phi(G)\}$, then $\rho\colon G \to F$ is an L-isomorphism of G into F. If $h \in G^+$ and $\omega \in \Phi(G)$, we define $h^\omega \in F$ by

$$(\phi)\, h^\omega = \begin{cases} h + P_\omega^* & \text{if } \phi = \omega, \\ P_\phi^* & \text{if } \phi \neq \omega, \end{cases}$$

and $H(h, \omega) \subseteq G$ by

$$H(h, \omega) = \{f \in G^+ \mid (\omega)(f\rho) \geq h + P_\omega^*\}.$$

LEMMA 3.2. *Let* G *be a completely distributive, representable* ℓ*-group that is not an* o*-group. Then, for all* $\omega \in \Phi(G)$, *and for all* $h \in B(G)^+$,

$$\bigwedge\nolimits_F [H(h, \omega)\rho] = h^\omega.$$

Proof. Clearly, $h^\omega \leq f\rho$ for all $f \in H(h, \omega)$. Conversely, suppose that $\ell \in F$ is such that $\ell \leq f\rho$ for all $f \in H(h, \omega)$. Since $h \in H(h, \omega)$,

$$\omega\ell \leq (\omega)(h\rho) = h + P_\omega^* = \omega h^\omega.$$

Let $\phi \in \Phi(G) \setminus \{\omega\}$. By Corollary 2.2, $B(G) = \langle P_\phi^* \cap B(G), P_\omega^* \cap B(G)\rangle$, and since G is representable, both $P_\phi^* \cap B(G)$ and $P_\omega^* \cap B(G)$ are normal in $B(G)$. Thus, since $h \in B(G)$,

$$h = k + t \quad \text{for some } k \in P_\phi^* \cap B(G) \text{ and some } t \in P_\omega^* \cap B(G).$$

Therefore

$$h + P_\omega^* = k + P_\omega^* = (\omega)(k\rho),$$

and hence $k \in H(h, \omega)$. Thus, $\ell \leq k\rho$, and hence

$$\phi\ell \leq (\phi)(k\rho) = P_\phi^* = \phi h^\omega.$$

Therefore, $\ell \leq h^{\omega}$, and the result follows.

Proof of Theorem 3.1. Let G be a completely distributive, representable ℓ-group that is not an o-group, and suppose that $h \in B(G)^{+} \setminus \{0\}$. Since G is completely distributive, $\bigcap \{ P_{\phi}^{*} \mid \phi \in \Phi(G)\} = \{0\}$ (see Section 1). Thus, $h \notin P_{\omega}^{*}$ for some $\omega \in \Phi(G)$. Suppose that $\bigwedge_G H(h, \omega) = 0$. Since ρ is an L-isomorphism, this implies by Lemma 3.2 that

$$h^{\omega} = \bigwedge_F [H(h, \omega)\rho] = \Big[\bigwedge_G H(h, \omega) \Big]\rho = 0\rho .$$

This contradicts our choice of h and ω, and hence there exists $s \in G$ such that $0 < s \leq f$ for all $f \in H(h, \omega)$. Therefore $0\rho < s\rho \leq f\rho$ for all $f \in H(h, \omega)$, and thus by Lemma 3.2,

$$0\rho < s\rho \leq \bigwedge_F [H(h, \omega)\rho] = h^{\omega} .$$

Since G/P_{ω}^{*} is totally ordered, $[0\rho, h^{\omega}]$ is totally ordered, and hence $[0\rho, s\rho]$ is totally ordered. Thus, since ρ is an ℓ-isomorphism, $[0, s]$ is totally ordered; that is, s is a basic element of G. Since $h \in H(h, \omega)$, $s \leq h$, and hence h exceeds a basic element. Therefore, B(G) has a basis.

PROPOSITION 3.3. *Let* G *be an ℓ-group. If* G *has a basis, then* $B(G)^{\perp} = \{0\}$. *Conversely, if* $B(G)^{\perp} = \{0\}$, *and if* G *is completely distributive and representable, then* G *has a basis.*

Proof. If G has a basis, then every $g > 0$ exceeds a basic element $s \in G$. By Proposition 2.3, $s \in B(G)$, and hence, since $s > 0$, $g \notin B(G)^{\perp}$. Therefore, $B(G)^{\perp} = \{0\}$. Conversely, suppose that G is completely distributive and representable and that $B(G)^{\perp} = \{0\}$. Then each $g \in G^{+} \setminus \{0\}$ is not in $B(G)^{\perp}$ and hence exceeds some $b \in B(G)^{+} \setminus \{0\}$. Since by Theorem 3.1, b exceeds a basic element of the convex ℓ-subgroup B(G), g exceeds a basic element of G. Therefore, G has a basis.

COROLLARY 3.4. *A completely distributive, representable ℓ-group* G *has a basis if and only if* $B(G)^{\perp} = \{0\}$.

4. EXAMPLES

Example 4.1. We noted in Section 2 that the bi-prime group can be very large in the cases of $R \,|\times|\, R$ and the group E of Example 2.4. Now, $R \,|\times|\, R$ is representable and completely distributive, and it has a basis. Since $B(R \,|\times|\, R) = R \,|\times|\, R$, the last two properties are derivable from each other via Theorem 3.1. The ℓ-group E is certainly representable but has no basic elements. Thus, by Theorem 3.1, since $B(E) = E$, E is not completely distributive.

Example 4.2 (see [9, p. 166, Example 5]). We construct a completely distributive, abelian (and hence representable) ℓ-group G that contains no basic elements and such that $P_{\phi} = \bigcap \{ \langle P_{\phi}, P_{\omega} \rangle \mid \omega \in \Phi(G) \setminus \{\phi\}\}$ for all $\phi \in \Phi(G)$.

Note. We shall use much of the terminology from [6] and [9] without explanation.

Let Λ be a root system in which each element covers exactly two elements and such that $\{x \in \Lambda \mid x \geq \lambda\}$ is finite for all $\lambda \in \Lambda$. Let $G = W(\Lambda, R)$. (That is, let G be the group $\sum \{R \mid \lambda \in \Lambda\}$ with lattice-order determined as follows: $f \geq 0$ in G if and only if $\lambda f \geq 0$ for all $\lambda \in \Lambda$ such that $\lambda f \neq 0$ but $\alpha f = 0$ for all $\alpha > \lambda$.) For each $g \in G^+ \setminus \{0\}$, it is easy to find $s, t \in G^+ \setminus \{0, g\}$ such that $s \vee t = g$ and $s \wedge t = 0$, and therefore G contains no basic elements. Let $\Gamma(G)$ ($\Gamma_1(G)$) be the set of all (regular) convex ℓ-subgroups of G. Then, by [6, Lemma 4.1], there exists a unique minimal plenary subset of $\Gamma_1(G)$, and hence by [9, Theorem 5.4] and [5, Theorem], G is completely distributive.

It remains to show that $P_\phi = T_\phi \equiv \bigcap \{\langle P_\phi, P_\omega \rangle \mid \omega \in \Phi(G) \setminus \{\phi\}\}$ for all $\phi \in \Phi(G)$. Note first that $B(G) = \{0\}$, by Theorem 3.1. By [6, Section 2], Λ may be considered as the set of completely meet-irreducible elements of the set Λ' of all dual ideals of Λ, and furthermore, in this case, Λ freely generates Λ'. Thus, by [6, Theorem 4.2], $\Gamma_1(G)$ is lattice-isomorphic to Λ and freely generates $\Gamma(G)$. Therefore $\{C \in \Gamma_1(G) \mid C \supseteq M\}$ is finite for all $M \in \Gamma_1(G)$, and hence every nonregular prime subgroup of G must be a minimal prime subgroup (see [8, p. 1.14, Theorem 1.7(7)]). Suppose now that $P_\phi \neq T_\phi$ for some $\phi \in \Phi(G)$. Then $T_\phi \in \Gamma_1(G)$, and hence there exists $H \in \Gamma_1(G)$ such that $P_\phi \subseteq H \subseteq T_\phi$ and $H \neq T_\phi$. Suppose that $T_\omega \subseteq H$ for some $\omega \in \Phi(G) \setminus \{\phi\}$. Then $P_\omega \subseteq T_\omega \subseteq H$, and thus

$$T_\phi \subseteq \langle P_\phi, P_\omega \rangle \subseteq H.$$

This contradicts our choice of H, and thus $T_\omega \not\subseteq H$ for all $\omega \in \Phi(G)$. Let

$$\Delta = \{F \in \Gamma_1(G) \mid T_\omega \subseteq F \text{ for some } \omega \in \Phi(G)\}.$$

Then $H \notin \Delta$. Clearly, Δ is a dual ideal of $\Gamma_1(G)$, and since each T_ω is prime, $T_\omega = \bigcap \{F \in \Delta \mid T_\omega \subseteq F\}$ (see [8, Theorem 1.7 (7)] again). Thus

$$\{0\} = B(G) = \bigcap \{T_\omega \mid \omega \in \Phi(G)\} = \bigcap \Delta.$$

Since $\Gamma_1(G)$ freely generates $\Gamma(G)$, $\Gamma_1(G)$ is itself the only dual ideal $\Xi \subseteq \Gamma_1(G)$ such that $\bigcap \Xi = \{0\}$, and therefore $\Delta = \Gamma_1(G)$. Since $H \notin \Delta$, this means that $H \notin \Gamma_1(G)$, which is a contradiction. We conclude that $P_\phi = T_\phi$ for all $\phi \in \Phi(G)$.

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