SEMIGROUPS WITH IDENTITY ON E^3

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Let M be a semigroup with identity on E^3, and let G be the maximal connected subgroup containing 1. It is well known that G is a three-dimensional Lie group and an open subset of M. In this paper, we show that if G has a nontrivial compact subgroup, then the boundary of G contains an idempotent. This result is a partial answer to a question posed by P. Mostert and A. Shields [13].

Let L be the boundary of G, and let S be the closed subsemigroup $G \cup L$. Any action of a subgroup of G on M, S, or L (an ideal of S) will be the obvious one via the semigroup multiplication in M. We assume that G contains a nontrivial compact subgroup C. It follows that C is isomorphic to the multiplicative group of complex numbers of norm one [12]. Also, each of the sets

$$F_1 = \{x \in M \mid xC = \{x\}\} \quad \text{and} \quad F_2 = \{x \in M \mid Cx = \{x\}\}$$

is a closed subset of M that is homeomorphic to E^1 [10]. If x is a point of M not in F_1, then xC, the right C-orbit through x, is homeomorphic to C. A similar statement is true regarding F_2 and left C-orbits. Because the closure of each G-orbit in L is a one-sided ideal in S, we may assume that no G-orbit in L is compact.

The following lemma implies that for each x in L,

$$\dim xG = 1 \Rightarrow x \in F_1 \cap L \quad \text{and} \quad \dim Gx = 1 \Rightarrow x \in F_2 \cap L.$$

Thus $x \in L \setminus (F_1 \cup F_2) \Rightarrow \dim Gx = \dim xG = 2$.

LEMMA 1. *If* xG *is a one-dimensional* G-*orbit in* L *that contains a subset* K *that is homeomorphic to a circle, then* xG = K.

Proof. Let P be a one-parameter subgroup of G such that xP = xG, and let $h: P \to xP$ be the map $h(p) = xp$. If h is not one-to-one, then $h(P) = K$. Suppose that h is one-to-one, and that $xP \neq K$. We shall reach a contradiction. The inverse of K under h cannot be compact. There exists a sequence $\{p_i\}$ in P such that $\{p_i\}$ has no convergent subsequence, such that for each i, $h(p_i) \in K$, and such that $h(p_i) \to k$ in K. Let $h(p) = k$, and let I be any finite closed interval about p in P. Clearly, h(I) is an arc with k in its interior, and $h(I) \cap K$ contains no subarc with k in its interior.

The P-orbit xP is locally homeomorphic to $Z \times A$, where Z is a zero-dimensional subset of xP, and A is an arc [7]. Thus we may assume that $Z \times A$ is a neighborhood of k in xP that contains an arc A_1 in K about k and an arc A_2 in h(I) about k. For $i = 1, 2$, the projection of A_i onto Z is a connected subset of Z containing k; hence A_1 and A_2 are both contained in the same fibre $\{k\} \times A$. This implies that $A_1 \cap A_2$ is an arc about k, contrary to the results of the previous paragraph.

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LEMMA 2. *If* L *contains no idempotent, then no point of* L *is a fixed point of* C *acting on the right or on the left in* L. *Thus for each* x *in* L, dim Gx = dim xG = 2.

Proof. Assume that L contains no idempotent. Since $G(F_1 \cap L) \subset F_1 \cap L$, it follows that dim Gx = 1 for each x in $F_1 \cap L$. The statement dual to Lemma 1 implies that $F_1 \cap L \subset F_2 \cap L$. A similar argument will show that $F_2 \cap L \subset F_1 \cap L$. Thus $F_1 \cap L = F_2 \cap L$, and we let F denote this set. F is an ideal in S; hence F is homeomorphic to a connected, closed subset of E^1. Since F cannot be compact, this implies that F is either a line, a half-line, or the empty set. The argument of [13, pp. 386-387] shows that there is an idempotent in F if F is a line or a half-line. Thus F is empty.

LEMMA 3. *For each* x *in* L, *let* P_x *denote the connected component containing* 1 *of* $G_r(x)$, *and assume that* L *contains no idempotent.*

(1) *If for each* x *and* y *in* L, P_x *is conjugate to* P_y, *then for each* x *in* L, $Gx \subset xG$, *and* G *has a normal one-parameter subgroup.*

(2) *If there exist an* x *in* L *and a closed two-dimensional subgroup* V *of* G *such that* dim xV = 2, *then* $Gx \subset xG$, *and* G *has a normal one-parameter subgroup.*

Proof. If for each x and y in L, P_x is conjugate to P_y, then by [3, page 315, Theorem 1.11], G has a local cross section at x. On the other hand, if there exists a closed two-dimensional subgroup V of G such that dim xV = 2, then V has no local isotropy at x, and by Theorem 1.8, page 312 (same reference), V has a local cross section at x. Consequently, the argument below for (2) is identical to that for (1).

There exist a neighborhood W of 1 in V and a closed subset D of L containing x such that $D \times W$ is homeomorphic to a neighborhood of x in L via the action of V on L. Since dim W = 2 = dim L, we know that dim D = 0 [5]. Thus D is totally disconnected. Let G_L denote the subgroup $\{g \in G \mid gx \in xG\}$ of G. If G_L contains a neighborhood of 1 in G, then $G_L = G$, and $Gx \subset xG$. This in turn will imply that $G_\ell(x)$ is normal, and the proof will be complete. Therefore let B be a closed ball about 1 in G such that $Bx \subset DW$. Since Bx is connected, it must be contained in a single fibre of DW. Thus $Bx \subset xW$, and $B \subset G_L$.

LEMMA 4. *If for each* x *in* L, *there exist closed two-dimensional subgroups* V_R *and* V_L *of* G *such that* dim xV_R = dim $V_L x$ = 2, *then* L *contains an idempotent.*

Proof. Let D be a local cross section to the action in L of V_R at x^2, and let N be a neighborhood of x in L such that $NN \subset DW$. There exist a y in N and an arc A in S such that the endpoints of A are 1 and y, and $A \cap L = \{y\}$ [2, p. 362]. It follows that yA is a connected, locally connected subset of L [14, p. 89], and thus there exists a connected subset U of DW containing some points in yG and the point y^2. The projection of U onto D is a connected subset of D containing y^2. Since D is totally disconnected (see the proof of Lemma 3), this implies that $y^2 \in yG$. It follows from Lemma 3 and its dual for left orbits that yG = Gy. This implies that yG is, algebraically, a group [4, p. 4]. Thus yG contains an idempotent.

THEOREM. *Let* M *be a semigroup with identity on* E^3, *and let* G *be the maximal connected subgroup containing* 1. *If* G *has a nontrivial compact subgroup, then the boundary of* G *contains an idempotent.*

Proof. Assume that L contains no idempotent. Then the conclusions of Lemma 2 hold. That is, for each x in L, dim Gx = dim xG = 2, $xC = \{x\}$, and $Cx = \{x\}$.

For each x in L, let P_x denote an arbitrary noncompact one-parameter subgroup of G that satisfies at least one of the two statements (1) $xP_x = \{x\}$, (2) $P_x x = \{x\}$. Let $(\tilde{G}, p)$ be a simply connected covering group of G. We denote the connected component containing $\tilde{1}$ of $p^{-1}(P_x)$ by $\tilde{P}_x$, and we denote the connected component containing $\tilde{1}$ of $p^{-1}(C)$ by $\tilde{C}$. The proof now proceeds in several steps that will, after some preliminaries, deal with the various possibilities for $\tilde{G}$ and arrive, in each case, at a contradiction, thus showing that our assumption is untenable.

We shall use the notation of [8], in referring to the various Lie groups on E^3. Basic information about these groups may be found on pages 309-310 of [8] and on pages 12-13 and 27-29 of [4]. There are two mistakes in these references that should be noticed. Since, in both [4] and [8], the term "semidirect product" refers only to those semidirect products on E^3 that have no center of positive dimension, the statement "$a \neq 1$" should be inserted beside the matrix (ii) on page 310 of [8]. (The semidirect product obtained by letting $a = 1$ is isomorphic to N.) On page 13 of [4] is a list of representations $t \to P(t)$ of the additive group of real numbers in the group of nonsingular 2-by-2 real matrices. The list is meant to be complete, but the possibility that $t \to P(t)$ might not be one-to-one is not treated. If $t \to P(t)$ is not one-to-one, then there exist a basis for E^2 and a nonzero real number θ such that for all real t,

$$P(t) = \begin{bmatrix} \cos t\theta & \sin t\theta \\ -\sin t\theta & \cos t\theta \end{bmatrix}.$$

The corresponding semidirect product on E^3 is isomorphic to the group

$$\begin{bmatrix} \cos t\theta & \sin t\theta & 0 & x \\ -\sin t\theta & \cos t\theta & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(1) *$\tilde{C}$ is a closed one-parameter subgroup of $\tilde{G}$, and the intersection $\tilde{P}_x \cap \tilde{C}$ is trivial.*

Argument. Since $p \mid \tilde{C}: \tilde{C} \to C$ is a covering, the first statement is clear. Similarly, $\tilde{P}_x$ is a closed one-parameter subgroup of $\tilde{G}$, and $p \mid \tilde{P}_x: \tilde{P}_x \to P_x$ is an isomorphism. Let $\tilde{H} = \tilde{P}_x \cap \tilde{C}$. Then $p(\tilde{H}) \subset P_x \cap C = \{1\}$. Thus $\tilde{H} = \{1\}$.

(2) *If $\tilde{G}$ is a semidirect product $V_2 R$, then $\tilde{G}$ (and hence G) has no normal one-parameter subgroup.*

Argument. Suppose that $(v, r) \in \ker p$ (which must be nontrivial). It is easy to verify that since (v, r) is in the center of G, the element v must be fixed under all inner automorphisms of G determined by elements of R, and the inner automorphism of G determined by r must fix all elements of V_2. An examination of the possibilities for $V_2 R$ shows that $\tilde{G}$ must be isomorphic to the group mentioned just above (1). Thus $\tilde{G}$ has no normal one-parameter subgroup. The center of $\tilde{G}$ is the infinite cyclic subgroup of R generated by $t = 2\pi/\theta$, and $\tilde{C} = R$. Notice, for later reference, that any two one-parameter subgroups of V_2 are conjugate by an inner automorphism of $\tilde{G}$ determined by an element of R.

(3) *If for each* $\tilde{P}_x$, *there exists a one-parameter subgroup* $\tilde{Q}$ *of* $\tilde{G}$ *such that* (i) *either* $\tilde{Q}$ *or* $\tilde{C}$ *is normal, and* (ii) *each element* $\tilde{g}$ *in* G *has a unique representation in the form* $\tilde{g} = \tilde{p}\tilde{q}\tilde{c}$, *where* $\tilde{p} \in \tilde{P}_x$, $\tilde{q} \in \tilde{Q}$, *and* $\tilde{c} \in \tilde{C}$, *then the hypotheses of Lemma* 4 *are satisfied.*

Argument. If $Q = p(\tilde{Q})$ were not closed, then Q^- would be a circle group, and that would contradict the fact that $p \mid Q$ is one-to-one. Thus Q is a closed one-parameter subgroup of G such that $Q \cap C$ is trivial. If either $\tilde{Q}$ or $\tilde{C}$ is normal, then QC is a closed two-dimensional subgroup of G. The hypotheses imply that each element $\tilde{g}$ in $\tilde{G}$ has a unique representation in the form $\tilde{g} = \tilde{c}\tilde{q}\tilde{p}$. Thus

$$x P_x = \{x\} \Rightarrow xG = xQC \quad \text{and} \quad P_x x = \{x\} \Rightarrow Gx = CQx = QCx.$$

Since $P_x \cap QC$ is trivial, the corresponding QC-orbit must be a two-dimensional subset of L.

Suppose A, B, and C are subgroups of a group D. In what follows, we shall write "D = ABC" in place of "each element d of D has a unique representation in the form d = abc, where $a \in A$, $b \in B$, and $c \in C$."

(4) $\tilde{G}$ *cannot be abelian or isomorphic to* $R \times Af(1)$.

Argument. We shall show that the hypotheses of (3) are satisfied. This is clear if $\tilde{G}$ is abelian; therefore we assume that $\tilde{G}$ is isomorphic to $R \times Af(1)$. Let T be the normal one-parameter subgroup of Af(1). If $\tilde{Q}$ is any one-parameter subgroup of $\tilde{G}$ different from T and from R, then $\tilde{G} = \tilde{Q}TR = T\tilde{Q}R$. The center of $\tilde{G}$ is R; therefore $R = \tilde{C}$. The hypotheses of (3) are satisfied.

(5) $\tilde{G}$ *cannot be isomorphic to the nonabelian nilpotent group* N.

Argument. As in (4), we shall show that $N = \tilde{P}_x\tilde{Q}\tilde{C}$. A representation of N is the group

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}.$$

$\tilde{C}$ is the center of $\tilde{G}$ and is the subgroup

$$\begin{bmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $t \to \tilde{P}(t)$ be a parametrization of $\tilde{P}_x$. Then

$$\tilde{P}(t) = \begin{bmatrix} 1 & x(t) & y(t) \\ 0 & 1 & z(t) \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easily verified that there exist real numbers a and b, not both zero, such that for all real t, $x(t) = a \cdot t$ and $y(t) = b \cdot t$. Suppose $a \neq 0$. (An argument similar to what follows will work if $a = 0$ and $b \neq 0$.) Given x, y, z, let $t = x/a$. There exists a unique $\bar{y}$ such that

$$\begin{bmatrix} 1 & x & y(t) \\ 0 & 1 & bt \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & z - bt \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \bar{y} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}.$$

It follows immediately that $\tilde{G} = \tilde{P}_x \tilde{Q} \tilde{C}$, where $\tilde{Q}$ is the subgroup

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}.$$

(6) $\tilde{G}$ *cannot be a semidirect product* $V_2 R$.

Argument. We have already seen that G is isomorphic to the group

$$\begin{bmatrix} \cos t\theta & \sin t\theta & 0 & x \\ -\sin t\theta & \cos t\theta & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and that $\tilde{C}$ is the subgroup

$$\begin{bmatrix} \cos t\theta & \sin t\theta \cdot & 0 & 0 \\ -\sin t\theta & \cos t\theta & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If some $\tilde{P}_x$ is not contained in V_2, then the dimension of the corresponding $p(V_2)$-orbit through x is two. This and part (2) of Lemma 3 yield a contradiction. If $\tilde{P}_x \subset V_2$ for each x in L, then $\tilde{P}_x$ is conjugate to $\tilde{P}_y$, for all x and y in L. This and part (1) of Lemma 3 yield a contradiction.

(7) G *cannot be isomorphic to the simply connected covering group* $S\ell(2)$ *of the group* $s\ell(2)$ *of* 2-by-2 *real matrices of determinant one.*

Argument. Let $q: \tilde{G} \to s\ell(2)$ be the covering map. Let K be the rotation group

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

and let W be the nonabelian planar subgroup of $s\ell(2)$ consisting of matrices of the form

$$\begin{bmatrix} t & b \\ 0 & 1/t \end{bmatrix},$$

where $t > 0$. It is easily verified directly that $s\ell(2) = KW$, and this implies that $\tilde{G} = \tilde{K}\tilde{W}$, where $\tilde{K}$ is any one-parameter subgroup of $\tilde{G}$ that contains the center, and $q \mid \tilde{W}: \tilde{W} \to W$ is an isomorphism. We may assume that $\tilde{K} = \tilde{C}$; therefore $G = CV$, where $p \mid \tilde{W}: \tilde{W} \to V$ is an isomorphism. Since G is a simple group, V is not normal; hence there exists a g in G such that $g^{-1} V g = U \neq V$. There exists at most a single one-parameter subgroup T of G such that $T \subset V \cap U$. If P_x is conjugate to T for each x in L, then Lemma 3 yields a contradiction. If P_x is not contained in V, for some x in L, then dim xV = 2, and Lemma 3 yields a contradiction. Suppose that $P_x \subset V$, for each $x \in L$, and that P_{x_0} is not conjugate to T. If $x_0 P_{x_0} = \{x_0\}$, then $P_{x_0 g} = g^{-1} P_{x_0} g$ is in U, but not in V, contrary to our assumption. A similar contradiction is reached if $P_{x_0} x_0 = \{x_0\}$.

We have considered each of the possibilities for $\tilde{G}$. In each case, the assumption that L contains no idempotent yields a contradiction. The proof is complete.

Comment. The theorem above is a partial answer, for the case n = 3, to a question posed by P. Mostert and A. Shields [13]. If M is a semigroup with identity on a connected (separable, metric) two-dimensional manifold, and G is the maximal connected subgroup containing 1, then, topologically, G^- is either (i) M, (ii) a plane, (iii) a half-plane, or (iv) the cartesian product of a half-line and a circle. This result, which answers completely the question referred to above for n = 2, does not (to my knowledge) appear in the literature, but it follows in a direct way if one exploits fully the following information: (a) the description above is valid for G^- if M is a plane [11], (b) the techniques and results in [6], (c) the simply connected covering space of a separable metric manifold is a separable metric manifold [9, page 181], and (d) the only simply connected (separable metric), noncompact, two-dimensional manifold is the plane [1, page 104]. The author is indebted to David Kahn, who was kind enough to point out that an arduous proof of this result was unnecessary.

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