HYPERINVARIANT SUBSPACES VIA TOPOLOGICAL PROPERTIES OF LATTICES

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1. INTRODUCTION

We denote by $\mathscr{H}$ a fixed, separable, infinite-dimensional, complex Hilbert space, and by $\mathscr{L}(\mathscr{H})$ the algebra of all (bounded, linear) operators on $\mathscr{H}$. A (closed) subspace of $\mathscr{H}$ will be said to be *hyperinvariant* for an operator T on $\mathscr{H}$ if it is an invariant subspace for every operator that commutes with T. The question whether every operator on $\mathscr{H}$ has a nontrivial hyperinvariant subspace is one of the most difficult problems in the theory of invariant subspaces.

Our principal objective in the present paper is to derive the existence of nontrivial hyperinvariant subspaces for operators whose lattices of invariant subspaces have a certain topological property (Theorem 2.2). Subsequently, we apply this result to prove that an operator whose invariant-subspace lattice satisfies a certain purely lattice-theoretic condition has a nontrivial hyperinvariant subspace (Theorem 3.1).

For each T in $\mathscr{L}(\mathscr{H})$, we shall denote by Lat T the lattice of invariant subspaces of T. We assume that Lat T is equipped with the (relative) topology induced by the following metric on the collection of all subspaces of $\mathscr{H}$. If $\mathscr{M}$ and $\mathscr{N}$ are subspaces, and $P_{\mathscr{M}}$ and $P_{\mathscr{N}}$ denote the (orthogonal) projections onto $\mathscr{M}$ and $\mathscr{N}$, respectively, then the distance between the subspaces $\mathscr{M}$ and $\mathscr{N}$ is defined by $\theta(\mathscr{M}, \mathscr{N}) = \|P_{\mathscr{M}} - P_{\mathscr{N}}\|$. The study of the topological properties of Lat T considered as a metric space under the metric θ was initiated in [2]. There it was proved that if $\mathscr{M}$ is an inaccessible point of Lat T (that is, if the arcwise connected component of $\mathscr{M}$ in Lat T is the singleton $\{\mathscr{M}\}$), then $\mathscr{M}$ is a hyperinvariant subspace for the operator T. In particular, if $\mathscr{M}$ is an isolated point of Lat T, then $\mathscr{M}$ is a hyperinvariant subspace for T. An interesting consequence of this is the following result proved in [5] by different methods, and involving only a lattice-theoretic condition. If $\mathscr{M}$ is a pinch point of Lat T (that is, if $0 \neq \mathscr{M} \neq \mathscr{H}$ and $\mathscr{M}$ is comparable with every subspace in Lat T), then $\mathscr{M}$ is a nontrivial hyperinvariant subspace of T (proof: $\theta(\mathscr{M}, \mathscr{N}) = 1$ for every $\mathscr{N}$ in Lat T different from $\mathscr{M}$). A generalization of this result appeared in [3] and reads as follows. If Λ is a countable subset of Lat T such that every $\mathscr{M}$ in (Lat T) $\setminus \Lambda$ is comparable with each subspace in Λ, then every $\mathscr{M}$ in Λ is a hyperinvariant subspace for the operator T. These results provide a point of departure for the present note, and as will be seen later, all of them are easy corollaries of our main result (Theorem 2.2).

The motivating idea of this note is to find interesting and useful topological conditions on the invariant-subspace lattice of an operator T in order to guarantee that T has nontrivial hyperinvariant subspaces. Since there may exist operators T in $\mathscr{L}(\mathscr{H})$ such that Lat T = $\{(0), \mathscr{H}\}$, one might question the utility of such hyperinvariant-subspace theorems; but it should be remembered that there are several interesting classes of operators (for example, the compact operators) for which the

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invariant-subspace problem is solved, but for which the hyperinvariant-subspace problem is open. Our present contribution is to the study of such classes of operators. For example, we show (Corollary 3.4) that if T is any operator such that Lat T is (lattice) isomorphic to the unit square, then T has a nontrivial hyperinvariant subspace.

2. A LOCAL PROPERTY

The following lemma, whose proof is given in [2], will be central to our purposes.

LEMMA 2.1. *Let* R *and* S *be invertible operators in* $\mathcal{L}(\mathcal{H})$, *and let* $\mathcal{M}$ *be a subspace of* $\mathcal{H}$. *Then*

$$\theta(R\mathcal{M}, S\mathcal{M}) \leq (\|R^{-1}\| + \|S^{-1}\|)\,\|R - S\| .$$

The following theorem, which is our main result, gives a sufficient condition of a local, topological nature that an operator have a nontrivial hyperinvariant subspace.

THEOREM 2.2. *Let* T *be an operator on* $\mathcal{H}$. *If* Ω *is an open subset of* Lat T, *and* Γ *is an arcwise connected component of* Lat T, *then the subspaces* $\mathcal{J} = \mathcal{J}(\Omega, \Gamma)$ *and* $\mathcal{K} = \mathcal{K}(\Omega, \Gamma)$ *defined by*

$$\mathcal{J} = \bigcap_{\mathcal{M}\in\Omega\cap\Gamma} \mathcal{M}, \qquad \mathcal{K} = \bigvee_{\mathcal{M}\in\Omega\cap\Gamma} \mathcal{M}$$

are hyperinvariant subspaces for T. *Thus, if there exist* Ω *and* Γ *such that either* $\mathcal{J}(\Omega, \Gamma)$ *or* $\mathcal{K}(\Omega, \Gamma)$ *is different from* (0) *and* $\mathcal{H}$, *then* T *has a nontrivial hyperinvariant subspace.*

Proof. Let S be any fixed nonzero operator in $\mathcal{L}(\mathcal{H})$ that commutes with T, and let $\mathcal{M}$ be any subspace in $\Omega \cap \Gamma$. To prove that $\mathcal{J} = \mathcal{J}(\Omega, \Gamma)$ and $\mathcal{K} = \mathcal{K}(\Omega, \Gamma)$ are hyperinvariant for T, it suffices to show that $S\mathcal{J} \subset \mathcal{M}$ and that $S\mathcal{M} \subset \mathcal{K}$, because this implies that

$$S\mathcal{J} \subset \bigcap_{\mathcal{M}\in\Omega\cap\Gamma} \mathcal{M} = \mathcal{J} \quad\text{and}\quad S\mathcal{K} = S\Big(\bigvee_{\mathcal{M}\in\Omega\cap\Gamma} \mathcal{M}\Big) \subset \bigvee_{\mathcal{M}\in\Omega\cap\Gamma} S\mathcal{M} \subset \mathcal{K} .$$

To this end, let λ be any complex number satisfying the condition $|\lambda| \leq 1/2\|S\|$, and observe that (by Lemma 2.1)

$$\begin{aligned}\theta(\mathcal{M}, (1 - \lambda S)\mathcal{M}) &\leq (1 + \|(1 - \lambda S)^{-1}\|)\,|\lambda|\,\|S\| \\ &\leq [1 + (1 - |\lambda|\,\|S\|)^{-1}]\,|\lambda|\,\|S\| \leq 3\,|\lambda|\,\|S\|\end{aligned}$$

and that

$$\theta(\mathcal{M}, (1 - \lambda S)^{-1}\mathcal{M}) \leq (1 + \|1 - \lambda S\|)\,\|(1 - \lambda S)^{-1}\|\,|\lambda|\,\|S\| \leq 5\,|\lambda|\,\|S\| .$$

For all sufficiently small ε, the subspaces $(1 - \varepsilon S)\mathcal{M}$ and $(1 - \varepsilon S)^{-1}\mathcal{M}$ clearly lie in $\Omega \cap \Gamma$. Thus, for such ε,

$$S\mathcal{J} \subset \mathcal{J} \vee (1 - \varepsilon S)\mathcal{J} \subset \mathcal{M} \vee (1 - \varepsilon S)(1 - \varepsilon S)^{-1}\mathcal{M} = \mathcal{M},$$

and $S\mathcal{M} \subset \mathcal{M} \vee (1 - \varepsilon S)\mathcal{M} \subset \mathcal{K}$. Since the last assertion of the theorem is obvious, the proof is complete.

COROLLARY 2.3. *If* Γ *is any arcwise connected component of* Lat T, *then the subspaces* $\bigvee_{\mathcal{M} \in \Gamma} \mathcal{M}$ *and* $\bigcap_{\mathcal{M} \in \Gamma} \mathcal{M}$ *are hyperinvariant subspaces for the operator* T.

COROLLARY 2.4 (Douglas and Pearcy). *If* $\mathcal{M}$ *is an inaccessible point of* Lat T, *then* $\mathcal{M}$ *is a hyperinvariant subspace for the operator* T.

THEOREM 2.5. *Let* T *belong to* $\mathcal{L}(\mathcal{H})$. *If* Λ *is a totally disconnected, open subset of* Lat T, *then every subspace in* Λ *is a hyperinvariant subspace for* T. *In particular, if* Λ *is a countable, open subset of* Lat T, *then each element of* Λ *is a hyperinvariant subspace for* T.

Proof. By virtue of Corollary 2.4, it is enough to prove that each subspace in Λ is an inaccessible point of Lat T. To this end, let γ be any continuous function from the closed interval $[0, 1]$ into Lat T such that $\gamma(0) \in \Lambda$. Also, let s be the supremum of the set

$$\{t \in [0, 1]\colon \gamma(r) = \gamma(0) \text{ whenever } 0 \leq r \leq t\}.$$

In order to prove that $\gamma(0)$ is an inaccessible point of Lat T, it clearly suffices to show that $s = 1$. If $s < 1$, then it follows from the definition of the number s, the continuity of the function γ, and the openness of the set Λ that there exists a positive number $\varepsilon \leq 1 - s$ such that $\gamma([s, s + \varepsilon]) \subset \Lambda$, and $\gamma(s) \neq \gamma(s + \varepsilon)$. This implies that the arcwise connected component of the point $\gamma(0)$ in the topological space Λ (with the topology induced by that of Lat T) is different from the singleton $\{\gamma(0)\}$, contradicting the fact that Λ is totally disconnected. Finally, recalling that Lat T is a complete metric space, we readily see that each countable open subset of Lat T is totally disconnected.

COROLLARY 2.6 (Fillmore, Rosenthal, and Stampfli). *Suppose that* Λ *is a countable subset of* Lat T *such that every subspace in* Λ *is comparable with each subspace in the complement of* Λ. *Then every subspace in* Λ *is hyperinvariant for* T.

Proof. If $\mathcal{M}$ and $\mathcal{N}$ are two subspaces of $\mathcal{H}$ such that $\mathcal{N} \subset \mathcal{M}$, then $\theta(\mathcal{M}, \mathcal{N}) = 1$. Thus $\theta(\mathcal{M}, \mathcal{N}) = 1$ for every $\mathcal{M}$ in Λ and every $\mathcal{N}$ in the complement of Λ. The desired conclusion follows from Theorem 2.5.

3. LATTICE-THEORETIC CONDITIONS

In this section, we give some lattice-theoretic conditions on the invariant-subspace lattice of an operator that are sufficient to ensure the existence of nontrivial hyperinvariant subspaces for the operator.

THEOREM 3.1. *Let* T *belong to* $\mathcal{L}(\mathcal{H})$, *and suppose that there exist two nonzero, proper subspaces* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *in* Lat T *such that* $\mathcal{M}_1 \subset \mathcal{M}_2$ *and such that every subspace in* Lat T *is comparable with either* $\mathcal{M}_1$ *or* $\mathcal{M}_2$. *Then* T *has a nontrivial hyperinvariant subspace.*

Proof. For $j = 1, 2$, let Ω_j be the open ball in Lat T with center at $\mathcal{M}_j$ and radius 1, and let Γ_j be the arcwise connected component of Lat T containing $\mathcal{M}_j$.

If $\mathcal{M} \in \Omega_1 \cap \Gamma_1$ and $\mathcal{M} \neq \mathcal{M}_1$, then $\mathcal{M}$ cannot be comparable with $\mathcal{M}_1$, and therefore $\mathcal{M}$ must be comparable with $\mathcal{M}_2$. If $\mathcal{M} \supset \mathcal{M}_2$, then $\mathcal{M} \supset \mathcal{M}_1$, which is impossible; therefore $\mathcal{M} \subset \mathcal{M}_2$. This shows that

$$\mathcal{K} = \bigvee_{\mathcal{M} \in \Omega_1 \cap \Gamma_1} \mathcal{M} \subset \mathcal{M}_2,$$

and a similar argument shows that

$$\mathcal{J} = \bigcap_{\mathcal{M} \in \Omega_2 \cap \Gamma_2} \mathcal{M} \supset \mathcal{M}_1 .$$

It follows immediately from Theorem 2.2 that $\mathcal{J}$ and $\mathcal{K}$ are nontrivial hyperinvariant subspaces for T.

COROLLARY 3.2. *Let* T *belong to* $\mathcal{L}(\mathcal{H})$, *and suppose that* Lat T *is isomorphic to the product of two complete lattices* Λ_1 *and* Λ_2, *one of which has a pinch point. Then* T *has a nontrivial hyperinvariant subspace.*

Proof. We recall that the ordering in the lattice $\Lambda_1 \times \Lambda_2$ is defined as follows. Let $\mathcal{J}_j$ and $\mathcal{K}_j$ be elements in Λ_j $(j = 1, 2)$. Then $(\mathcal{J}_1, \mathcal{K}_1) \leq (\mathcal{J}_2, \mathcal{K}_2)$ if and only if $\mathcal{J}_1 \leq \mathcal{J}_2$ and $\mathcal{K}_1 \leq \mathcal{K}_2$. Suppose, without loss of generality, that Λ_1 contains a pinch point $\mathcal{N}$, and set $\mathcal{M}_1 = (\mathcal{N}, 0)$ and $\mathcal{M}_2 = (\mathcal{N}, 1)$. Then $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfy the hypotheses of Theorem 3.1, and hence T has a nontrivial hyperinvariant subspace.

COROLLARY 3.3. *If the lattice of an operator* T *is isomorphic to the product of a complete lattice and a nontrivial chain, then* T *has a nontrivial hyperinvariant subspace.*

The following corollary provides a concrete application of the preceding results.

COROLLARY 3.4. *If the lattice of an operator* T *in* $\mathcal{L}(\mathcal{H})$ *is* (*lattice-*) *isomorphic to the* (*solid*) *unit square, then* T *has a nontrivial hyperinvariant subspace.*

Examples of operators whose lattice of invariant subspaces is isomorphic to the unit square are easy to obtain. If A and B are operators on $\mathcal{H}$ such that the spectrum of A is "very disjoint" from the spectrum of B, and both Lat A and Lat B are isomorphic to the unit interval, then Lat $(A \oplus B)$ is isomorphic to the unit square [2, Theorem 6]. A less trivial example of an operator whose invariant subspace lattice is the unit square is given in [6]. The operator discussed there is $T = \sqrt{V} \oplus -\sqrt{V}$, where V is the Volterra integral operator. Unfortunately, the existence of a nontrivial hyperinvariant subspace for these examples can be deduced without recourse to Corollary 3.4. In fact, it follows from [1, Theorem 2.4] that the direct sum of two operators has a nontrivial hyperinvariant subspace whenever one of them has this property. Thus it would be of interest to find an example of an operator T on $\mathcal{H}$ such that Lat T is the unit square and such that T has no nontrivial complemented invariant subspaces. However, it is not completely clear that such an operator T exists, because if the algebraic sum of the subspaces $\mathcal{M}$ and $\mathcal{N}$ in Lat T corresponding to (0, 1) and (1, 0), respectively, were closed, then $\mathcal{M}$ and $\mathcal{N}$ would be complementary invariant subspaces for T. Whether $\mathcal{M} + \mathcal{N}$ is necessarily closed we are at present unable to say.

4. EXTENSIONS AND OPEN QUESTIONS

Perusal of the preceding proofs shows that the results stated in Sections 2 and 3 are valid in a considerably more general context. It is easy to see that the Hilbert-space structure plays no role, so that $\mathscr{H}$ might as well have been taken to be a complex Banach space $\mathscr{X}$. Furthermore, we could have replaced the single operator T with an arbitrary algebra $\mathscr{A}$ of operators on $\mathscr{X}$. The definition of Lat $\mathscr{A}$ is then obvious, as well as what it means to say that a subspace $\mathscr{M}$ of $\mathscr{X}$ is hyperinvariant for $\mathscr{A}$. It is not obvious what function should play the role of the metric θ, but it turns out (see [4]) that there is a suitable metric $\tilde{\theta}$, called the "gap between subspaces", with which one can topologize Lat $\mathscr{A}$. (By use of this metric, most of the results of [2] were extended to complex Banach spaces, in [4].)

Using the results of [4], one sees easily that the results in Sections 2 and 3 of the present note remain valid with $\mathscr{H}$ and T replaced by $\mathscr{X}$ and $\mathscr{A}$, and that the proofs remain the same. We leave to the interested reader the task of formulating the more general results and verifying the validity of the proofs.

It would be interesting to consider "familar-looking" lattices L and to ask whether there exists an operator T in $\mathscr{L}(\mathscr{H})$ such that Lat T = L, and in case the answer is affirmative, whether T has a nontrivial hyperinvariant subspace. Suppose, for example, that L is the (solid) triangle with the vertices (0, 0), (1, 0), and (1, 1). In this case, we do not know the answer to the first question; but an easy application of Theorem 3.1 shows that if there exists an operator T with Lat T = L, then T has a nontrivial hyperinvariant subspace. What happens if L is the union of the diagonals of the unit square?

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