UNIVERSALLY COMMUTATABLE OPERATORS ARE SCALARS

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1. INTRODUCTION

Let A and B be finite-dimensional linear operators that generate one-parameter groups $P_t = e^{tA}$ and $Q_t = e^{tB}$. Then the groups generated by A + B and AB - BA can be expressed by means of the well-known Lie product formulas

$$\lim_{n \to \infty} (P_{t/n} Q_{t/n})^n = \exp t(A + B), \tag{1}$$

$$\lim_{n \to \infty} (P_{-\sqrt{t/n}}\, Q_{-\sqrt{t/n}}\, P_{\sqrt{t/n}}\, Q_{\sqrt{t-n}})^n = \exp t(AB - BA). \tag{2}$$

An infinite-dimensional version of (1) was proved by H. F. Trotter [9]. It states that (1) is valid if P_t and Q_t are (C_0) contraction semigroups on a Banach space such that the closure $[A + B]^-$ of the sum of their generators itself generates a (C_0) semigroup R_t. The right side of (1) is to be interpreted as R_t, and the limit is in the strong operator topology. In [1], the present author proved a rather general theorem that includes Trotter's. J. A. Goldstein [5] and E. Nelson [8, Theorem 8.7] have used this result to prove infinite-dimensional versions of the commutator formula (2).

The limits in (1) and (2) may exist even when the hypotheses of [5], [8], and [9] are not satisfied. By our general theory (see [2], [3]) the limits must be semigroups, if they exist at all. If they are (C_0) semigroups, we denote their generators by $A +_L B$ and $[A, B]_L$. The subscript L refers to a generalized Lie operation. These generalized operations can be quite pathological. A detailed study of generalized addition of self-adjoint operators is contained in [3]; a number of examples concerning both addition and commutation can be found in [6]. In particular, we showed in [3] that only *bounded* self-adjoint operators A can be added--by the Lie process or by any other reasonable process—to every self-adjoint operator B. In fact, if A is not bounded, then one can construct a B such that the symmetric operator A + B, defined on $\mathscr{D}(A) \cap \mathscr{D}(B)$, has no self-adjoint extensions.

Goldstein [6] has conjectured that an analogous situation holds for commutators. Let $\mathscr{H}$ be an infinite-dimensional Hilbert space. Call a self-adjoint operator A *universally commutatable in the classical sense* if for all self-adjoint B the operator AB - BA, defined on $\mathscr{D}(AB) \cap \mathscr{D}(BA)$, is essentially skew-adjoint; call A *universally commutatable in the Lie sense* if $[A, B]_L$ exists for all self-adjoint B. The only operators that are obviously universally commutatable (in either case) are the scalar multiples of the identity. As we shall show, there are no other operators universally commutatable in the classical sense or the Lie sense, at least if the definition of the latter is strengthened in a technical way.

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To be precise, in Section 2 we shall establish the following result: if A is a nonscalar self-adjoint operator on an infinite-dimensional space, then there exists a self-adjoint B such that AB - BA is not closable, in other words, such that the closure of AB - BA is not even an operator, much less a skew-adjoint operator. This immediately takes care of the classical case. In Section 3, we give an improved product formula for commutators; it enables us to apply the result of Section 2 to the case of operators that are universally commutatable in the Lie sense (see Section 4).

2. A CHARACTERIZATION OF SCALARS

Let A be a self-adjoint operator on a Hilbert space $\mathscr{H}$. We shall say that A has *property* S provided that for every self-adjoint operator B on $\mathscr{H}$, the operator AB - BA, defined on $\mathscr{D}(AB) \cap \mathscr{D}(BA)$, is closable. Scalar multiples of the identity have property S; we shall prove the converse.

2.1. THEOREM. *Let* A *be a self-adjoint operator on an infinite-dimensional Hilbert space. If* A *has property* S, *then* A *is a scalar multiple of the identity.*

The proof will be accomplished through a sequence of lemmas.

2.2. LEMMA. *If* A *has property* S, *then so has every direct summand of* A. *Likewise, so has* $A - \lambda I$ *for each scalar* λ.

Proof. By a direct summand A' we mean the restriction of A to one of its invariant subspaces $\mathscr{K}$. Given B' on $\mathscr{K}$, we can extend it to B on $\mathscr{H}$ by defining $B = 0$ on $\mathscr{K}^{\perp}$. Then AB - BA is a closable extension of A'B' - B'A', so that the latter is closable. Thus A' has property S.

As for $A - \lambda I$, it has property S because $[A - \lambda I, B] = [A, B]$. (Here equality includes equality of domains.) ■

2.3. LEMMA. *If* A *has property* S, *then* A *is bounded.*

Proof. If A is not bounded, let e be a unit vector not in $\mathscr{D}(A)$. Let P denote orthogonal projection onto the span of e. Then $C = [A, P]$ is defined on $\{e\}^{\perp} \cap \mathscr{D}(A)$, and on this domain $Cx = -PAx = -(Ax, e)e$. Because $e \notin \mathscr{D}(A)$, there is a sequence in $\mathscr{D}(A)$ with $x_n \to 0$ but $(Ax_n, e) \to 1$. Since $\mathscr{D}(A)$ is dense, we can find $z \in \mathscr{D}(A)$ with $(z, e) = 1$. Let $y_n = x_n - (x_n, e)z$. Then y_n is orthogonal to e, $y_n \in \mathscr{D}(A)$, $y_n \to 0$, and

$$-Cy_n = (Ay_n, e)e = (Ax_n, e)e - (x_n, e)(Az, e)e - e \neq 0.$$

Thus C is not closable; that is, A does not have property S. ■

We now make a construction that will repeatedly be useful. Let $\mathscr{K}$ be a separable Hilbert space, and let T be a fixed unbounded self-adjoint operator on $\mathscr{K}$. Pick a unit vector v_0 ($v_0 \notin \mathscr{D}(T)$). Since $\mathscr{D}(T)$ is dense in $\{v_0\}^{\perp}$, we can extend v_0 to an orthonormal basis $\{v_0, v_1, v_2, \cdots\}$, where $v_n \in \mathscr{D}(T)$ if $n > 0$. Define $K_n = \|Tv_n\|$ $(n = 1, 2, \cdots)$.

Suppose $e_0, e_1, e_2, \cdots$ is an orthonormal sequence in another Hilbert space $\mathscr{H}$. Define a self-adjoint operator B by setting $B = 0$ on the complement of the span of the elements e_n, and $B = U^{-1}TU$ on the span of the elements e_n, where $Ue_n = v_n$. Thus $e_0 \notin \mathscr{D}(B)$, $e_n \in \mathscr{D}(B)$ if $n > 0$, $Bx = 0$ if x is orthogonal to every e_n, and $\|Be_n\| = K_n$ if $n > 0$. We shall call B the *standard operator* associated with the sequence $\{e_n\}_0^{\infty}$.

2.4. LEMMA. *If* A *has property* S, *then* A *has no infinite-dimensional, nonzero, compact direct summand.*

Proof. By Lemma 2.2, we can assume that A itself is compact. Choose eigenvectors $e_0, e_1, e_2, \cdots$ for A with corresponding eigenvalues $\lambda_0, \lambda_1, \lambda_2, \cdots$ such that $\lambda_0 \neq 0$ and $\sum_{n=1}^{\infty} |\lambda_n| K_n < \infty$. Here the constants K_n are defined as in the preceding construction.

Let B be the standard operator associated with the sequence $\{e_n\}_0^\infty$. We shall show that [A, B] is not closable. First note that $Ae_0 = \lambda_0 e_0$ is not in $\mathscr{D}(B)$. On the other hand, $A(\{e_0\}^\perp)$ is contained in $\mathscr{D}(B)$. Indeed, a typical vector orthogonal to e_0 is

$$x = \sum_{n=1}^{\infty} \xi_n e_n + y,$$

where y is orthogonal to each of the vectors e_n. Then

$$Ax = \sum_{n=1}^{\infty} \xi_n \lambda_n e_n + Ay,$$

and formally,

$$BAx \sim \sum_{n=1}^{\infty} \xi_n \lambda_n Be_n .$$

(Note that BAy = 0.) Now the formal series actually converges in norm, by the choice of the scalars λ_n. Therefore Ax is in $\mathscr{D}(B)$, because B is a closed operator. Conclusion: $\mathscr{D}(BA) = \{e_0\}^\perp$, and BA is a bounded operator, by the closed-graph theorem.

Therefore [A, B] = AB - BA, with domain $\mathscr{D} = \mathscr{D}(B) \cap \{e_0\}^\perp$. Because BA is bounded, it suffices to show that AB is not closable on $\mathscr{D}$. To see this, choose $y_n \in \mathscr{D}$, as in the proof of Lemma 2.3, so that $y_n \to 0$ but $(By_n, e_0) \to 1$. Let $u = \alpha e_0 + x$ (with $(x, e_0) = 0$) be an arbitrary vector in $\mathscr{H}$. Then

$$\begin{aligned}(ABy_n, u) &= (By_n, Au) = \lambda_0 \alpha(By_n, e_0) + (By_n, Ax) \\ &= \lambda_0 \alpha(By_n, e_0) + (y_n, BAx) \to \lambda_0 \alpha + 0 = (\lambda_0 e_0, u).\end{aligned}$$

That is, $ABy_n \to \lambda_0 e_0$ in the weak topology. This shows that AB is not closable. ■

We can now see that if A has property S and has a basis of eigenvectors, then A is a scalar. Indeed, if A has infinitely many distinct eigenvalues, they have an accumulation point λ. The operator $A - \lambda$ then has an infinite-dimensional, nonzero, compact summand, contrary to Lemma 2.4. Hence A has only finitely many eigenvalues. One of these, say λ, has infinite multiplicity. If $A \neq \lambda I$, then $A - \lambda$ has an infinite-dimensional, nonzero, compact summand, and we obtain the same contradiction.

The remainder of the argument is devoted to showing that an operator with property S does indeed have a basis of eigenvectors.

2.5. LEMMA. *Suppose* A *is a nonzero, bounded, self-adjoint operator with purely continuous spectrum. Let* $\{\alpha_n\}_0^\infty$ *be any sequence of positive numbers. Then there exist a real number* λ *and two orthonormal sequences* $\{e_n\}_0^\infty$ *and* $\{f_n\}_0^\infty$ *such that*

(1) $(e_m, f_n) = 0$ *for all* m *and* n,

(2) $(A - \lambda)e_n = \lambda_n f_n$ *for* $n = 0, 1, 2, \cdots$, *where* $0 < |\lambda_n| \leq \alpha_n$.

Proof. The spectrum of A is an infinite closed subset of $\mathbb{R}$ without isolated points, hence it is uncountable. Let λ be any point of the spectrum other than an endpoint of one of its countably many complementary intervals. By replacing A by $A - \lambda$, we may assume that $\lambda = 0$. Then, for each $\varepsilon > 0$, both $(0, \varepsilon)$ and $(-\varepsilon, 0)$ contain points of the spectrum of A.

Hence we can choose infinitely many disjoint, measurable subsets of $\mathbb{R}$, say E_n^+, E_n^- $(n = 0, 1, 2, \cdots)$, such that the corresponding spectral projections are nonzero, and such that

$$E_n^+ \subseteq (0, \alpha_n), \qquad E_n^- \subseteq (-\alpha_n, 0).$$

Define $E_n = E_n^+ \cup E_n^-$, and let P_n be the spectral projection of E_n.

Since A is both positive and negative on the range of P_n, we can choose a unit vector e_n in this range such that $(Ae_n, e_n) = 0$. Define $f_n = Ae_n/\|Ae_n\|$. Then $\{f_n\}_0^\infty$ is an orthonormal sequence, $(e_m, f_n) = 0$ for each m, and $Ae_n = \lambda_n f_n$, where $\lambda_n = \|Ae_n\| \leq \|P_n A P_n\| \leq \alpha_n$, by construction. ∎

2.6. LEMMA. *If* A *has property* S, *then* A *has a basis of eigenvectors.*

Proof. Write A as $A' \oplus A''$, where A' has a basis of eigenvectors and A'' has purely continuous spectrum. We assert that $A'' = 0$. Otherwise, by passing to a summand, we may assume that $A = A''$. We shall deduce a contradiction.

If $A = A''$, let $\{e_n\}_0^\infty$ and $\{f_n\}_0^\infty$ be orthonormal sequences, as in the proof of Lemma 2.5. We may assume that the number λ is 0, so that $Ae_n = \lambda_n f_n$, with $0 < |\lambda_n| \leq \alpha_n$. Here we choose α_n so that $\sum_{n=1}^\infty |\lambda_n| K_n < \infty$. Now let B be the standard operator associated with the sequence $\{e_n\}_0^\infty$. We shall show that [A, B] is not closable.

Let $\mathcal{M}$ denote the closed span of the elements e_n. Note that if $x \in \mathcal{M}$, then $Ax \in \mathcal{M}^\perp$, so that $BAx = 0$. Hence, on $\mathcal{M} \cap \mathcal{D}(B)$ we have the relation $[A, B]x = ABx$. Moreover, on $\mathcal{M}$ we can write $A = VC$, where $Ce_n = \lambda_n e_n$ and V is any isometry such that $Ve_n = f_n$ for all n. By arguing as in the final paragraph of the proof of Lemma 2.4, we see that CB is not closable on the domain $\mathcal{M} \cap \mathcal{D}(B)$. But, on this domain, $AB = VCB$. Thus AB is not closable, and A fails to have property S. ∎

It follows immediately from Theorem 2.1 that every self-adjoint operator on an infinite-dimensional Hilbert space that is universally commutatable in the classical sense must be a scalar, since such operators obviously have property S. Actually, a much shorter proof of this result could be given. If A is one-to-one with dense range, then by a result of von Neumann (see [4, Theorem 3.6]), there exists a B with $\mathcal{D}(B) \cap \mathcal{R}(A) = (0)$, so that [A, B] is defined only at 0 and is therefore very far indeed from being essentially skew-adjoint. Using this result, together with the analogue of Lemma 2.2, one can easily show that a classically universally commutatable A has only finitely many points in its spectrum. By an argument like that in the proof of Lemma 2.3 (but reversing the roles of A and P), one can deduce that A is a scalar. We shall need our more involved argument with its more powerful conclusion, in order to deal with the case of universally commutatable operators in the Lie sense.

3. A PRODUCT FORMULA FOR COMMUTATORS

3.1. THEOREM. *Let* e^{tA} *and* e^{tB} *be* (C_0) *one-parameter groups of isometries on a Banach space. Assume that* A *is bounded, and that the closure* C *of* AB - BA *is the generator of a* (C_0) *semigroup. Then for each* $t \geq 0$ *and each vector* x, *we have the relation*

$$\lim_{n\to\infty} \left(e^{-\sqrt{t/n}\,A} e^{-\sqrt{t/n}\,B} e^{\sqrt{t/n}\,A} e^{\sqrt{t/n}\,B}\right)^n x = e^{tC} x. \tag{1}$$

The convergence is uniform on compact t-*intervals.*

Proof. Define $F(t) = e^{-\sqrt{t}\,A} e^{-\sqrt{t}\,B} e^{\sqrt{t}\,A} e^{\sqrt{t}\,B}$. We shall apply our general product theorem [2]. Note that each F(t) is a contraction. We shall show that the strong derivative F'(0) is an extension of AB - BA.

We can write

$$F(t^2) = e^{-tA} e^{tA_t}, \tag{2}$$

where $A_t = e^{-tB} A e^{tB}$ is a bounded operator; in fact, $\|A_t\| = \|A\|$. Expanding (2) in power series, we see that

$$F(t^2) = I + t(A_t - A) + \frac{t^2}{2}(A^2 - 2AA_t + A_t^2) + O(t^3). \tag{3}$$

Note that A_t converges to A in the strong operator topology as $t \to 0$. Therefore the coefficient of t^2 in (3) tends strongly to 0.

Now suppose that $u \in \mathscr{D}(B) \cap \mathscr{D}(BA)$. Then

$$\begin{aligned}(A_t - A)u &= e^{-tB}(A e^{tB} - e^{tB} A)u \\ &= e^{-tB}(Au + tABu - Au - tBAu + o(t)) = t e^{-tB}[A, B]u + o(t).\end{aligned}$$

Substituting this in (3), we obtain the relation

$$(F(t^2) - I)u = t^2 e^{-tB}[A, B]u + o(t^2). \tag{4}$$

It follows that $\lim_{t\to 0} \frac{1}{t^2}[F(t^2) - I]u = [A, B]u.$ ■

Theorem 3.1 is in some respects an improvement of Nelson's result [8, Theorem 8.7], which (for operators on Hilbert spaces) requires that [A, B] be essentially skew-adjoint on $\mathscr{D}(AB) \cap \mathscr{D}(BA) \cap \mathscr{D}(A^2) \cap \mathscr{D}(B^2)$. When A is bounded, this reduces to $\mathscr{D}(B^2) \cap \mathscr{D}(BA)$, which is smaller than our domain $\mathscr{D}(B) \cap \mathscr{D}(BA)$. It is easy to find examples in which our condition is satisfied but Nelson's fails to hold.

The following technical result will be needed in Section 4.

3.2. PROPOSITION. *As in Theorem* 3.1, *let* e^{tA} *and* e^{tB} *be* (C_0) *one-parameter groups of isometries on a Banach space, with* A *bounded. Suppose that for each vector* x *the limit*

$$R_t x = \lim_{n\to\infty} (e^{-\sqrt{t/n}A} e^{-\sqrt{t/n}B} e^{\sqrt{t/n}A} e^{\sqrt{t/n}B})^n x$$

exists, uniformly for t *in compact intervals. Then* $R_t = e^{tC}$ *is a* (C_0) *one-parameter semigroup, and* C *is an extension of* [A, B].

Proof. That R_t is a (C_0) semigroup follows from general theory [3, Section 2]. It can also be seen directly, under the assumption of uniform convergence. This assumption also implies, by [3, Theorem 3.7], that C is an extension of the strong derivative F'(0). Here F(t) is defined as in the proof of Theorem 3.1, and the argument above shows that F'(0) is itself an extension of [A, B]. ∎

4. UNIVERSALLY COMMUTATABLE OPERATORS

We have already seen that operators universally commutatable in the classical sense must be scalars. Our methods are not quite sufficient to obtain this result for operators universally commutatable in the Lie sense as previously defined. However, we can get a slightly weaker result if we strengthen the definition by adding a uniformity condition.

Definition. A self-adjoint operator A is *universally commutatable in the strong Lie sense* provided that for each self-adjoint B the products

$$(e^{-i\sqrt{t/n}A} e^{-i\sqrt{t/n}B} e^{i\sqrt{t/n}A} e^{i\sqrt{t/n}B})^n \tag{1}$$

converge in the strong operator topology to a semigroup e^{tC}, uniformly on compact t-intervals.

4.1. THEOREM. *If* A *is universally commutatable in the strong Lie sense on an infinite-dimensional space, then* A *is a scalar multiple of the identity.*

Proof. First, we claim that A is bounded. If not, let B be a projection of rank 1, as in Lemma 2.3, such that [A, B] is not closable. Proposition 3.2 implies that if (1) converges as in the definition, then the generator C extends [A, B]. Since C is closed, this is a contradiction.

Knowing that A is bounded, we see immediately that it has property S. Indeed, if B is any self-adjoint operator, the same reasoning shows that [A, B] has a closed extension C.

Hence A is a scalar, by Theorem 2.1. ∎

The imposition of uniformity in Theorem 4.1, while important from a technical point of view, seems relatively innocuous. Its only purpose is to guarantee that $[A, B]_L$ be an extension of AB - BA. Surely, any reasonable definition ought to meet this requirement. We note that uniformity follows automatically from mere convergence [3, Theorem 3.1] if AB - BA happens to be densely defined.

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