THE MODULI OF EXTREMAL FUNCTIONS

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Let D be a domain on the Riemann sphere that supports nonconstant bounded analytic functions, and let p be a point in D. The extremal problem of maximizing $|f'(p)|$ over the class of functions f that are holomorphic and bounded by 1 in D is known to have a solution unique up to multiplication by unimodular constants (see [2]); the solution with positive derivative is called the *Ahlfors function* for D and p. Since both D and p are fixed, I shall suppress them in the notation and denote the Ahlors function for p and D by $\Phi(z)$. It is known that $\Phi(p) = 0$. It is also known that if D is bounded by a finite number of disjoint, analytic, simple closed curves, then Φ is analytic on the boundary of D and has unit modulus there. This implies that for a domain of this type, Φ has unit modulus on the Silov boundary of the Banach algebra $H^\infty(D)$. For a general domain, it makes no sense to talk about the boundary values of a bounded holomorphic function; but it does make sense to discuss the values of the (transform of this) function on the maximal ideal space of $H^\infty(D)$. The main result of this note is that for a general domain, the Ahlfors function for D and p has unit modulus on the Silov boundary of $H^\infty(D)$. The main result and another result on the modulus of the Ahlfors function are in Section 1; Section 2 contains some related matters, extensions, and open problems concerning the Ahlfors function.

1. THE MODULUS OF THE AHLFORS FUNCTION

THEOREM 1. *Let* Φ *be the Ahlfors function for* D *and* p. *Then* (*the Gelfand transform of*) Φ *has unit modulus on the Silov boundary of* $H^\infty(D)$; *equivalently, for each* $h \in H^\infty(D)$, $\|h\| = \|\Phi h\|$.

Proof. Let Ω consist of all points w in D for which there exists an $h \in H^\infty(D)$ such that $|h(w)| > 1$ and $\|h\Phi\| \leq 1$. If Ω is empty (as I wish to show), then $\|h\| = \|h\Phi\|$ for each $h \in H^\infty(D)$, as desired. Hence, to reach a contradiction, I assume that Ω is not empty. Clearly, Ω is an open subset of D. I shall show that Ω is also a closed subset of D; since D is connected, this will imply that $\Omega = D$. Hence, there exists an $h \in H^\infty(D)$ with $|h(p)| > 1$ and $\|h\Phi\| \leq 1$. Thus, $|(h\Phi)'(p)| = |h(p)|\,\Phi'(p) > \Phi'(p)$, while $\|h\Phi\| \leq 1$; this contradicts the extremal property of Φ. Thus the remainder of the proof is devoted to showing that Ω is a closed subset of D (equivalently, that $D - \Omega$ is open).

Let $r \in D - \Omega$, and let $\{z_i\}$ be a dominating sequence for $H^\infty(D)$; that is, let $\sup |h(z_i)| = \|h\|$ for every $h \in H^\infty(D)$; see [6] for a discussion of dominating sequences. There is no loss in assuming that $r \notin \{z_i\}$. Let M be the maximal ideal space of ℓ^∞. If F is a bounded function defined on a neighborhood of $\{z_i\}$, the restriction of F to $\{z_i\}$ gives an element of ℓ^∞; I shall denote the transform of such an element by $\hat{F}$.

Received September 21, 1971.
This research was supported in part by NSF grant GP-19526.

Michigan Math. J. 19 (1972).

Since $r \notin \Omega$, we know that the condition $\|\Phi h\| \leq 1$ implies that $|h(r)| \leq 1$ for every $h \in H^\infty(D)$. Hence, there is a measure m on M of total variation at most 1, with

$$\int_M \hat{h}\hat{\Phi}\, dm = h(r) \qquad \text{for all } h \in H^\infty(D).$$

Thus $1 = \int \hat{1}\hat{\Phi}\, dm = \int \hat{\Phi}\, dm \leq \int |\hat{\Phi}|\, d|m| \leq \|\Phi\|\, \|m\| \leq 1$, and therefore the measure $d\rho = \hat{\Phi}\, dm$ is positive and has mass 1, and its closed support lies in the set where $|\hat{\Phi}| = 1$; also,

$$\int_M \hat{h}\, d\rho = h(r) \qquad \text{for all } h \in H^\infty(D).$$

Now let $p(z; s) = \frac{z - r}{z - s}$ for s close to r. Then, since $F(s) = \int_M \hat{p}(\ ; s)\, d\rho$ is a continuous function of s for s near r and has the value 1 when $s = r$, it is nonzero in a neighborhood of r. Let s be a point near r where $F(s) \neq 0$. For $h \in H^\infty(D)$, let

$$g(z) = (h(z) - h(s))\frac{z - r}{z - s}.$$

Then $g \in H^\infty(D)$ and $\int_M \hat{g}\, d\rho = 0$. Hence

$$\int_M \hat{h}\hat{p}(\ ; s)\, d\rho = h(s)\, F(s),$$

so that the measure $d\beta = (F(s))^{-1}\hat{p}(\ ; s)\, d\rho$ represents evaluation at s for $H^\infty(D)$. Hence, if $\|\hat{h}\|_K \leq 1$, where $K = \{|\hat{\Phi}| = 1\}$, then $|h(s)| \leq C$, where $C = \|\beta\|$ is independent of h. Replace h by h^n, then take nth roots, and let n approach infinity. The condition $\|\hat{h}\|_K \leq 1$ implies that $|h(s)| \leq 1$ for all $h \in H^\infty(D)$. In particular, $\|\Phi h\| \leq 1$ implies $|h(s)| \leq 1$, and hence $s \notin \Omega$. Thus $D - \Omega$ is open; equivalently, Ω is closed in D. As I outlined above, this leads to a contradiction.

Definition. A point $\xi \in \partial D$ is *removable* if for each $h \in H^\infty(D)$ there exists some neighborhood U of ξ, which may depend on h, such that h has a holomorphic extension to U. Otherwise, ξ is *essential.* If each point in ∂D is essential, then D is *maximal.*

COROLLARY. *Let $\xi \in \partial D$. Then ξ is essential if and only if*

$$1 = \limsup \{|\Phi(z)| : z \in D \text{ and } z \to \xi\}.$$

Proof. Suppose ξ is essential but $\limsup |\Phi(z)| = 1 - \delta$, where $\delta > 0$. By a theorem of A. Beck [1], there exists a function $h \in H^\infty(D)$ with

$$\limsup\{|h(z)| : z \to \xi\} = 1 \quad \text{and} \quad \limsup\{|h(z)| : z \to \lambda\} < 1$$

$$\text{for every } \lambda \in \partial D - \xi.$$

Thus $\lim\sup\{|\Phi(z)h(z)|: z \to \lambda\} < 1$ for every $\lambda \in \partial D$, and hence $\|\Phi h\| < 1$. But $\|\Phi h\| = \|h\| = 1$, by Theorem 1.

Conversely, suppose ξ is removable. Then there exists a domain D^* containing D and ξ, and such that every function in $H^\infty(D)$ extends to a function in $H^\infty(D^*)$ (see [6]). Should it happen that $\lim\sup\{|\Phi(z)|: z \to \xi\} = 1$, then $|\Phi|$ would have an interior maximum in D^*, and hence Φ would be a constant, a contradiction.

The proposition that Φ has modulus 1 on the Silov boundary of $H^\infty(D)$ may constitute the best possible result. Nevertheless, a more concrete result would be preferable, especially since the Silov boundary of $H^\infty(D)$ is difficult to visualize. One possibility for a more appealing theorem might be found along these lines: let F be the uniformizer of D; then the composition $\Phi \circ F$ is a bounded holomorphic function on the open unit disc Δ, and as such it has radial boundary values on a set of full measure on the unit circle T; in the case where D is bounded by a finite number of disjoint, nontrivial continua, $\Phi \circ F$ is an inner function; that is, $|\Phi \circ F| = 1$ a.e. $(d\theta)$ on T. As a first step, one might guess that this is true in general; however, it is easy to find a counterexample. For example, let us form D by deleting from Δ a compact set K of positive logarithmic capacity but analytic capacity 0 (see [8]). Then $\Phi(z) = z$ (assuming $p = 0$), but the uniformizer F cannot be an inner function, for this would require that its range omits only a set of logarithmic capacity 0; but $\log \operatorname{cap}(K) > 0$. There is an obvious weakness in this example, however: the domain D is not maximal for $H^\infty(D)$; that is, ∂D includes removable singularities. Hence, we are left with the following problem.

QUESTION 1. *Let* D *be a maximal domain, and let* F *be its uniformizer. Is* $\Phi \circ F$ *an inner function? If not, is there at least a set of positive measure on which the boundary values of* $\Phi \circ F$ *have modulus* 1?

Of course, at any point where ∂D is "nice," Φ is continuous and has modulus 1 (see [2, Theorem 5]) and if in some sense most of ∂D is nice, then $\Phi \circ F$ is an inner function (see [2, Theorem 7]). The problem is to deal with domains having very bad boundaries. L. A. Rubel and J. Ryff [4] have dealt with this problem and have obtained results about the modulus of $\Phi \circ F$ for some special types of domains. Another such result is Theorem 2 below; in order to present it, I first give some background information.

In dealing with multiply connected domains, it is natural to consider certain multiple-valued bounded analytic functions. To be precise, let F be a bounded, multiple-valued, holomorphic function on D whose modulus is single-valued, and let γ be a smooth closed curve in D. Then continuation of a function element of F around γ results in multiplication by a constant of absolute value 1 (since $|F|$ is single-valued). We denote this constant by $\Gamma_F(\gamma)$ and note that $\Gamma_F(\gamma)$ is the same on homotopic curves, and is independent of the point on γ from which the continuation is begun. Thus $\Gamma_F(\gamma)$ is a character on the fundamental group $\pi(D)$ of D. The solutions of various function-theoretic problems concerning multiply connected domains depend on the fact that if Γ is a character on $\pi(D)$, then there exists a bounded, multiple-valued, holomorphic function F on D whose modulus is single-valued, with $\Gamma_F = \Gamma$. H. Widom [7] discusses this, and he gives necessary and sufficient conditions for the existence of such a function. To state Widom's theorem, we need some notation. If Γ is a character on $\pi(D)$, let $\mathscr{H}^\infty(D, \Gamma)$ be the set of bounded, multiple-valued, holomorphic functions F on D whose modulus is single-valued and for which $\Gamma_F = \Gamma$. If $\zeta \in D$, let

$$M(D, \Gamma, \zeta) = \sup\{|F(\zeta)|: F \in \mathscr{H}^\infty(D, \Gamma) \text{ and } |F| \leq 1 \text{ in } D\}$$

and

$$M(D, \zeta) = \inf\{M(D, \Gamma, \zeta): \Gamma \text{ is a character on } \pi(D)\}.$$

If $\mathscr{H}^\infty(D, \Gamma)$ is empty, we set $M(D, \Gamma, \zeta) = 0$.

THEOREM (Widom [7]). *A necessary and sufficient condition that each* $\mathscr{H}^\infty(D, \Gamma)$ *be nonempty is that* $M(D, \zeta) > 0$ *for some (and hence all)* $\zeta \in D$.

We shall use this theorem to prove the following result.

THEOREM 2. *Suppose* $M(D, \zeta) > 0$ *for some* $\zeta \in D$. *Then* $\Phi \circ F$ *is an inner function.*

Proof. Suppose there exists a set E on the unit circle T, of positive measure, on which $\Phi \circ F$ has modulus less than $1 - \delta$ for some δ $(1/2 > \delta > 0)$. There is no loss in assuming that E is invariant under the group G of conformal self-maps α of Δ that satisfy the condition $F \circ \alpha = F$ (see [3]). Hence u, the harmonic extension to Δ of the characteristic function of E, is also invariant under G, and thus there exists a positive harmonic function v on D with $u(z) = v(F(z))$ for all $z \in \Delta$. Let c be given by the formula $c(1 - \delta/e)e = 1$, and let $f(z) = c(\exp(u + i\,{}^*u))$, where ${}^*u$ is the harmonic conjugate of u on Δ. Then f is a bounded holomorphic function on Δ, and

$$|f(z)| = \begin{cases} (1 - \delta/e)^{-1} & (z \in E), \\ c < 1 & (z \in T - E). \end{cases}$$

Note also that $|(f)(\Phi \circ F)| \leq 1$ a.e. on T. Further, $|f|$ is invariant under the group G, and hence there exists a multiple-valued, bounded, holomorphic function $A(z)$ on D with $A(F(z)) = f(z)$ for all $z \in \Delta$. Choose a point r in Δ with $|f(r)| > 1 + \varepsilon$ $(\varepsilon > 0)$. Widom's theorem implies that for each n there exists a multiple-valued holomorphic function h_n on D such that $|h_n| \leq 1$, such that $|h_n(F(r))| \geq \nu > 0$, and such that $g_n = A^n h_n$ is a single-valued holomorphic function and ν does not depend on n. Note that

$$\|g_n\| \geq |A(F(r))|^n \, |h_n(F(r))| \geq (1 + \varepsilon)^n \nu \to \infty$$

as $n \to \infty$. However, since multiplication by Φ is an isometry of $H^\infty(D)$, we have for each n the relations

$$\|g_n\| = \|\Phi^n g_n\|_D = \|(\Phi^n \circ F)(f^n)(h_n \circ F)\|_\Delta \leq \|(\Phi \circ F)(f)\|^n \leq 1.$$

This is a contradiction; hence, $\Phi \circ F$ must be an inner function.

Remarks. Once we know that $\Phi \circ F$ is an inner function on Δ, we may ask whether it has a singular factor. In the case where D is bounded by a finite number of nontrivial continua, $\Phi \circ F$ is an (infinite) Blaschke product. Whether this is true when D is maximal for H^∞ is unknown.

2. RELATED MATTERS AND OPEN PROBLEMS

A topic closely related to the modulus of Φ is the range of Φ. It is known that if ∂D consists of n disjoint, nontrivial continua, then Φ maps D exactly n-to-1 onto Δ. Again, the example cited in Section 1 shows that in the general case Φ may omit many points in Δ (but the omitted set always has analytic capacity 0 (see [2, Theorem 3]). If Φ is an inner function, then the omitted set has logarithmic capacity zero. Using the theory of cluster sets, I can show that in certain special situations the mapping Φ covers each point of Δ infinitely often, with perhaps one exception. In the general case, nothing as strong is known about the range of Φ.

QUESTION 2. *Let* D *be a maximal domain. Does* Φ *map* D *onto* Δ? *If* D *is infinitely connected, does each point of* Δ *have infinitely many inverse images?*

Finally, the reader should note that the proof of the Theorem in Section 1 uses only a few properties of $H^\infty(D)$, namely

(1) $H^\infty(D)$ is an algebra that contains the constants,

(2) if $\|f_n\| \leq C$, $f_n \in H^\infty(D)$, and $f_n(z) \to f(z)$ for each $z \in D$, then $f \in H^\infty(D)$, and

(3) if $f \in H^\infty(D)$ and $f(\alpha) = 0$ for some $\alpha \in D$, then $(z - \alpha)^{-1} f(z) \in H^\infty(D)$.

If A is a subalgebra of $H^\infty(D)$ satisfying (1), (2), and (3), then there exists a unique function Φ in A with $\Phi'(p) \geq \max\{|f'(p)| : f \in A, \|f\| < 1\}$, and for this Φ, $|\Phi| = 1$ on the Silov boundary of A. In general, many proper subalgebras A of $H^\infty(D)$ satisfy (1), (2), and (3), even if D is simply connected. We can obtain such a subalgebra by taking for D the open unit disc Δ with the set $\{x: 0 \leq x < 1\}$ deleted and letting A be the restriction to D of the functions that are analytic and bounded on the complement of $[0, 1/2]$ relative to the Riemann sphere.

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