DENSITY THEOREMS FOR ALGEBRAS OF OPERATORS AND ANNIHILATOR BANACH ALGEBRAS

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INTRODUCTION

Let X be a Banach space, $\mathscr{B}(X)$ the algebra of bounded operators on X, and $\mathscr{F}(X)$ the algebra of bounded operators on X with finite-dimensional range. An algebra A of operators on X is *irreducible* if the only closed A-invariant subspaces of X are $\{0\}$ and X (many authors use "transitive" in place of "irreducible"). In Section 1, we consider conditions on a subalgebra A of $\mathscr{B}(X)$ under which $\mathscr{F}(X) \subset A$. In particular, we show that if X is reflexive and A is a closed irreducible subalgebra of $\mathscr{B}(X)$ that contains a nonzero operator in $\mathscr{F}(X)$, then $\mathscr{F}(X) \subset A$ (Theorem 2).

In Section 2, we prove some Stone-Weierstrass theorems in the setting where B is an annihilator algebra and A is a closed subalgebra satisfying certain conditions (Theorems 9, 10, and 11). The following is a special case of Theorem 10: If B is an annihilator B^*-algebra and A is a semisimple closed subalgebra of B with the property that for each pair of distinct maximal left ideals M and N in B, A contains an element in M but not in N, then A = B. In [7], I. Kaplansky proved a similar theorem with the stronger hypothesis that A is a closed *-subalgebra of B (see [7, Theorem 2.2, p. 223]).

1. CONDITIONS IMPLYING THAT A SUBALGEBRA OF $\mathscr{B}(X)$ CONTAINS $\mathscr{F}(X)$

An algebra A is called *semiprime* if A has no nonzero nilpotent left or right ideals. In fact, A is semiprime if it has no nonzero nilpotent left ideals. For in this case, suppose that R is a nonzero right ideal of A such that $R^n = \{0\}$ and $R^{n-1} \neq \{0\}$, where n is an integer $(n > 1)$. Set $N = R^{n-1}$. Then $N \neq \{0\}$ and $N^2 = \{0\}$. Choose $a \in N$ $(a \neq 0)$. Then $(aA)^2 \subset N^2 = \{0\}$. Therefore $(Aa)^3 = A(aA)^2 a = \{0\}$. By hypothesis, a belongs to the set $\{b \in A \mid Ab = \{0\}\}$; but this set is a nilpotent left ideal of A. This contradiction proves that A has no nonzero nilpotent right ideals.

Throughout this paper, X denotes a Banach space of dimension greater than 1.

LEMMA 1. *Let* A *be an irreducible subalgebra of* $\mathscr{B}(X)$. *Then* A *is semiprime.*

Proof. By the remarks preceding the lemma, it suffices to prove that A has no nonzero nilpotent left ideal. Suppose A contains such a left ideal. Then there exists $T \in A$ $(T \neq 0)$ such that $(AT)^2 = \{0\}$. The set

$$Y = \{x \in X \mid Sx = 0 \text{ for all } S \in A\}$$

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is a proper, closed, A-invariant subspace of X, and therefore $Y = \{0\}$. But $TAT(X) \subset Y$, so that $TAT = \{0\}$. Choose an $x \in X$ such that $T(x) \neq 0$. Then the A-invariant subspace $AT(x)$ is dense in X and is contained in the null space of T. This contradiction proves the lemma.

Let $\mathscr{D}(X)$ be the set consisting of the zero operator and all the operators in $\mathscr{B}(X)$ with 1-dimensional range. Let X^* be the normed dual space of X. If $x \in X$ and $f \in X^*$, we can define $T \in \mathscr{D}(X)$ by the condition $T(y) = f(y) \cdot x$ $(y \in X)$. We use the notation $(f \mid x)$ for this operator T. Conversely, it is easy to verify that if $T \in \mathscr{D}(X)$, then $T = (f \mid x)$ for some $x \in X$ and some $f \in X^*$.

THEOREM 2. *Let* A *be a subalgebra of* $\mathscr{B}(X)$ *with the properties*

(i) A *acts irreducibly on* X,

(ii) $\mathscr{D}(X) \cap A$ *is closed in* $\mathscr{B}(X)$,

(iii) $\mathscr{F}(X) \cap A \neq \{0\}$.

Then there exists a closed subspace $K \subset X^*$ *such that*

(1) $(g \mid y) \in A$ *whenever* $g \in K$ *and* $y \in X$,

(2) $\{y \mid g(y) = 0$ *for all* $g \in K\} = \{0\}$.

In particular, if X *is reflexive, then* $\mathscr{F}(X) \subset A$.

Proof. First we prove that A contains a minimal idempotent. By (iii), $\mathscr{F}(X) \cap A$ is a nonzero ideal of A. Then, by Lemma 1, there exist $T, S \in \mathscr{F}(X) \cap A$ such that $ST \neq 0$. The algebra TAS is finite-dimensional. If there exists a positive integer n such that $(TAS)^n = \{0\}$, then the ideal $(AST)^{n+1} = \{0\}$. This contradicts Lemma 1. It follows by Wedderburn Theory (see [6, pp. 38, 53, 54]) that TAS is not a radical algebra, and consequently there exists a nonzero idempotent $F \in TAS$. Suppose that the finite-dimensional algebra FAF has a nonzero, nilpotent left ideal. Then there exists $T \in FAF$ $(T \neq 0)$ such that $(FAFT)^2 = \{0\}$. Note that $T = FT = TF$. Then, whenever $Q, R, S \in A$, we have the relation

$$QTRTST = Q(FTRF)T(FSF)T = \{0\}.$$

This implies that $(AT)^3 = \{0\}$, a contradiction. It follows from Wedderburn Theory that FAF is semisimple. Therefore there exists a projection $E \in FAF$ $(E \neq 0)$ such that $EFAFE = \{\lambda E \mid \lambda \text{ complex}\}$. Then, for each $S \in A$, there exists a scalar λ such that $ESE = EFSFE = \lambda E$. Thus E is a minimal idempotent of A.

Now let $K = \{g \in X^* \mid (g \mid y) \in A \text{ for some } y \in X,\ y \neq 0\}$. The proof of [4, Theorem 3] implies that K has properties (1) and (2).

When X is reflexive, then (2) implies that $K = X^*$. By (1), it follows that $\mathscr{D}(X) \subset A$. Therefore, since $\mathscr{F}(X)$ is the linear span of $\mathscr{D}(X)$, we see that $\mathscr{F}(X) \subset A$.

As a corollary of Theorem 2, we have a result of E. A. Nordgren, R. Radjavi, and P. Rosenthal (see [9, Corollary 2, p. 177]).

COROLLARY 3. *If* A *is an irreducible, weakly closed subalgebra of* $\mathscr{B}(X)$ *and* $\mathscr{F}(X) \cap A \neq \{0\}$, *then* $A = \mathscr{B}(X)$.

Proof. Theorem 2 applies to A, and therefore there exists a subspace K of X^* with properties (1) and (2). Property (2) implies that K is dense in X^* in the X-topology. Therefore, if $g \in X^*$, there exists a net $\{g_\alpha\} \subset K$ such that $g_\alpha(y) \to g(y)$ for every $y \in X$. If $x \in X$, the net of operators $(g_\alpha \mid x)$ converges weakly to $(g \mid x)$. By (1), $\{(g_\alpha \mid x)\} \subset A$, and therefore $(g \mid x) \in A$. This implies that $\mathscr{D}(X) \subset A$. It follows that $\mathscr{F}(X) \subset A$. It is easy to verify that $\mathscr{F}(X)$ is weakly dense in $\mathscr{B}(X)$, and this proves the corollary.

We say that an algebra $A \subset \mathscr{B}(X)$ *separates the independent vectors of* X provided to each pair of (linearly) independent vectors $x, y \in X$ there corresponds an operator $T \in A$ such that $Tx = 0$ and $Ty \neq 0$.

LEMMA 4. *Suppose* A *is a semiprime subalgebra of* $\mathscr{B}(X)$ *that separates the independent vectors of* X, *and* E *is a minimal idempotent of* A; *then* $E = (f \mid x)$ *for some* $x \in X$ *and some* $f \in X^*$.

Proof. Suppose x and y are independent in X, and $Ex = x$ and $Ey = y$. By hypothesis, there exists $T \in A$ such that $Tx = 0$ and $Ty \neq 0$. Clearly, $TE \neq 0$, and it follows from [10, Corollary (2.1.19), p. 46] that $ATE = AE$. Therefore there exists $S \in A$ such that $STE = E$. But then $x = Ex = STx = 0$. This contradiction proves that $E \in \mathscr{D}(X)$, and this implies that $E = (f \mid x)$ for some $x \in X$, $f \in X^*$.

Let T^* be the adjoint (or conjugate) operator of an operator $T \in \mathscr{B}(X)$. When $x \in X$ and $f \in X^*$, we sometimes use the notation $\langle x, f \rangle$ for $f(x)$. In this notation, $\langle Tx, f \rangle = \langle x, T^* f \rangle$ whenever $x \in X$, $f \in X^*$, and $T \in \mathscr{B}(X)$. Corresponding to subspaces $\mathscr{J} \subset X$ and $\mathscr{K} \subset X^*$, we write

$$\mathscr{J}^{\perp} = \{f \in X^* \mid \langle x, f \rangle = 0 \text{ for all } x \in \mathscr{J}\},$$

$$^{\perp}\mathscr{K} = \{x \in X \mid \langle x, f \rangle = 0 \text{ for all } f \in \mathscr{K}\}.$$

THEOREM 5. *Assume that* X *is reflexive, and that* A *is a semiprime subalgebra of* $\mathscr{B}(X)$ *such that*

(i) $\mathscr{F}(X) \cap A \neq \{0\}$,

(ii) A *separates the independent vectors of* X *and* $A^* = \{T^* \mid T \in A\}$ *separates the independent vectors of* X^*.

Then $\mathscr{F}(X) \subset \overline{A}$.

Proof. As in the proof of Theorem 2, A contains a minimal idempotent E. By Lemma 4, $E = (f \mid x)$ for some $x \in X$, $f \in X^*$. Set $\mathscr{J} = \{Tx \mid T \in A\}$ and $\mathscr{K} = \{T^* f \mid T \in A\}$. The subspace $\mathscr{J}$ is A-invariant, and $\mathscr{K}$ is A^*-invariant. Therefore $^{\perp}\mathscr{K}$ is A-invariant.

Next we prove that $\mathscr{J} \cap {}^{\perp}\mathscr{K} = \{0\}$. Because $E^2 = E$, a simple computation shows that $f(x) = 1$. Also, $Ex = (f \mid x)x = f(x) \cdot x = x$. Now assume that $y \in \mathscr{J} \cap {}^{\perp}\mathscr{K}$. Then $y = Tx$ for some $T \in A$. If $g \in X^*$ and $S \in A$, then

$$S^* E^*(g) = g(x) \cdot S^*(f) \in \mathscr{K}.$$

Since $Tx \in {}^{\perp}\mathscr{K}$, $\langle ESTEx, g \rangle = \langle Tx, S^* E^* g \rangle = 0$. It follows that $ESTEx = 0$. There exists a scalar λ such that $ESTE = \lambda E$. The relations $\lambda x = \lambda Ex = ESTEx = 0$ imply that $\lambda = 0$, so that $ESTE = 0$ for all $S \in A$. But then $\{R \in AE \mid EAR = \{0\}\}$ is a nilpotent left ideal of A and contains TE. Therefore $y = Tx = TEx = 0$. This establishes that $\mathscr{J} \cap {}^{\perp}\mathscr{K} = \{0\}$. Suppose $y \in {}^{\perp}\mathscr{K}$ and $y \neq 0$. Then $x + y$ and $x - y$ are independent vectors in X. But if $T \in A$ and $T(x + y) = 0$, then

$$T(x) = -T(y) \in \mathscr{J} \cap {}^{\perp}\mathscr{K} = \{0\}.$$

Thus $Tx = T(y) = 0$, so that $T(x - y) = 0$. This contradicts (ii). It follows that $^{\perp}\mathscr{K} = \{0\}$. Since X is reflexive, $\overline{\mathscr{K}} = X^*$.

Both $\mathscr{K}$ and $\mathscr{J}^{\perp}$ are A^*-invariant. A proof similar to the one above establishes that $\mathscr{K} \cap \mathscr{J}^{\perp} = \{0\}$. Therefore $\mathscr{J}^{\perp} = \{0\}$, and consequently, $\overline{\mathscr{J}} = X$. Given $y \in X$ and $g \in X^*$, we can choose $\{y_n\} \subset \mathscr{J}$ and $\{g_n\} \subset \mathscr{K}$ so that $y_n \to y$ and $g_n \to g$. Then there exist sequences $\{T_n\}, \{S_n\} \subset A$ such that $T_n x = y_n$ and $S_n^* f = g_n$ for all n. Note that $(g_n \mid y_n) = T_n(f \mid x)S_n \in A$ for all n, and $(g_n \mid y_n) \to (g \mid y)$ in the operator norm. Therefore $(g \mid y) \in \overline{A}$. This proves that $\mathscr{D}(X) \subset \overline{A}$, and it follows that $\mathscr{F}(X) \subset \overline{A}$.

2. STONE-WEIERSTRASS THEOREMS FOR ANNIHILATOR ALGEBRAS

Let B be a semisimple annihilator Banach algebra. F. F. Bonsall and A. W. Goldie proved in [5] that if e is a minimal idempotent of B, then Be is a reflexive Banach space. They showed that eB can be identified with the dual space of Be, in the sense that for each $a \in eB$ there exists a unique $f_a \in (Be)^*$ such that

$$abe = eabe = f_a(be)\cdot e \qquad \text{for all } be \in Be.$$

The map $a \to f_a$ is a bicontinuous isomorphism of eB onto $(Be)^*$ (see [5, p. 161] for details). Together with the reflexivity of Be, the identification of eB with $(Be)^*$ leads to the following result.

LEMMA 6. *Let* B *and* e *satisfy the hypotheses above. Then*

(i) *if* H *is a proper closed subspace of* Be, *there exists* $b \in eB$ $(b \neq 0)$ *such that* $ba = 0$ *for all* $a \in H$;

(ii) *if* H *is a proper closed subspace of* eB, *there exists* $b \in Be$ $(b \neq 0)$ *such that* $ab = 0$ *for all* $a \in H$.

Let A be a subalgebra of an algebra B. We say that A *separates the maximal left ideals of* B if whenever M and N are distinct maximal left ideals of B, then there exists an element of A in M that is not in N. In [7], Kaplansky uses this kind of separation property in his generalization of the Stone-Weierstrass Theorem to certain B^*-algebras. Now we prove a proposition that is basic to the results of this section.

PROPOSITION 7. *Let* B *be a semisimple annihilator Banach algebra. Assume that* A *is a closed semisimple subalgebra of* B *and that* A *separates the maximal left ideals of* B. *If* e *is a minimal idempotent of* A, *then* e *is a minimal idempotent of* B, $eA = eB$, *and* $Ae = Be$.

Proof. Let e be a minimal idempotent of A. Then $A(1 - e)$ is a maximal left ideal of A. We prove first that e is a minimal idempotent of B. For suppose it is not. Then, by [3, Theorem 2.2, p. 497], there exist g and f, minimal idempotents of B, such that $fg = 0$, $ef = f$, and $eg = g$. Therefore $B(1 - f)$ and $B(1 - g)$ are distinct maximal left ideals of B, and

$$A(1 - e) \subset A \cap B(1 - f), \qquad A(1 - e) \subset A \cap B(1 - g).$$

But then $A(1 - e) = A \cap B(1 - f) = A \cap B(1 - g)$. This contradiction proves that e is a minimal idempotent of B.

The set $J = \{be \mid eAbe = \{0\},\ b \in B\}$ is a subspace of Be, and it is closed under left multiplication by elements in A. Therefore $Ae \cap J$ is a left ideal of A that does not contain e. Therefore $Ae \cap J = \{0\}$. Corresponding to $c \in Be$ $(c \neq 0)$, set $M_c = \{b \in B \mid bc = 0\}$. By [10, p. 45], $cB = fB$ for some minimal idempotent f of B. The set $M_c = B(1 - f)$ is a maximal left ideal of B. Now suppose that $a \in J$ and $a \neq 0$. Then M_{e+a} and M_{e-a} are distinct maximal left ideals of B. But if $b \in A \cap M_{e+a}$, then $be = -ba \in Ae \cap J = \{0\}$. Therefore $b \in A \cap M_{e-a}$. This contradiction proves that $J = \{0\}$. Now it follows from Lemma 6 that $eA = eB$.

If $a \in A$ and $ea \neq 0$, then $eaA = eA$. Therefore

$$\{ea \mid ea \in A,\ eaAe = \{0\}\} = \{0\}.$$

Since $eA = eB$, it follows further that

$$\{eb \mid eb \in B,\ ebAe = \{0\}\} = \{0\}.$$

By Lemma 6, $Ae = Be$.

COROLLARY 8. *Let* X *be a reflexive Banach space. Assume that* B *is a Banach algebra, that* $\mathscr{F}(X) \subset B \subset \mathscr{B}(X)$, *and that* $\mathscr{F}(X)$ *is dense in* B *in the Banach algebra norm on* B. *If* A *is a closed semisimple subalgebra of* B *that separates the independent vectors of* X, *then* $A = B$.

Proof. By [10, pp. 102-104], B is an annihilator algebra. For each $x \in X$, set $M_x = \{T \in B \mid Tx = 0\}$. Every maximal left ideal of B is of the form M_x, for some $x \in X$ $(x \neq 0)$. Since A separates the independent vectors of X, A separates the maximal left ideals of B. Furthermore, A contains a minimal idempotent E, by [2, Theorem 2.2, p. 512]. Hence, BEB is a nonzero ideal in the topologically simple algebra B. Therefore $AEA = BEB$ is dense in B. Since A is closed, $A = B$.

We denote the socle of an algebra C by S_C.

THEOREM 9. *Let* B *be a semisimple annihilator Banach algebra. Assume that* A *is a closed, semisimple subalgebra of* B, *that* S_A *is dense in* A, *and that* A *separates the maximal left ideals of* B. *Then* $A = B$.

Proof. The proof of [5, Theorem 5, p. 158] shows that AeA is a minimal (two-sided) ideal of A whenever e is a minimal idempotent of A. Furthermore, the proof of [5, Theorem 6, p. 158] shows that A is the closure of the sum of the minimal ideals AeA. For each minimal idempotent e of A, $AeA = BeB$, by Proposition 7. Therefore A is a closed ideal of B. By [1, Proposition 3.2, p. 567], either $S_B \subset A$ or there exists a minimal idempotent $f \in B$ such that $Af = \{0\}$. But if $Af = \{0\}$, then A is contained in the maximal left ideal $B(1 - f)$. This is impossible, so that $S_B \subset A$. Since S_B is dense in B, it follows that $A = B$.

It was assumed that Theorem 9 that S_A is dense in A. If B is topologically simple, then this hypothesis is unnecessary. For in this case there is a faithful representation of B as an algebra of operators on a reflexive Banach space having the properties of the algebra B in the statement of Corollary 8 (see [10, pp. 102-104]). Corollary 8 then yields the result without the assumption that S_A is dense in A. We do not know whether this hypothesis can be dropped from Theorem 9 in general. We shall now verify that if B is a B^*-algebra, the hypothesis can be dropped. Given a nonempty subset J of the algebra B, let $\mathscr{R}[J] = \{b \in B \mid Jb = \{0\}\}$. If J is a closed (two-sided) ideal of B, then $\mathscr{R}[J]$ is a closed ideal of B. Also, if B is an annihilator B^*-algebra, then $B = J \oplus \mathscr{R}[J]$, by [5, Theorem 3, p. 157].

THEOREM 10. *Let* B *be a semisimple annihilator Banach algebra with the special property that* $B = J \oplus \mathscr{R}[J]$ *whenever* J *is a closed ideal of* B. *If* A *is a closed semisimple subalgebra of* B *that separates the maximal left ideals of* B, *then* A = B.

Proof. A is a modular annihilator algebra, by [2, Corollary, p. 517]. We prove that S_A is dense in A, and then the theorem follows from Theorem 8. Let J be the closure of the sum of the ideals of the form AeA, where e is a minimal idempotent of A. Then J is the closure of S_A, and as in the proof of Theorem 8, J is a closed ideal of B. By [1, Theorem 4.2 (2), p. 569], $A \cap \mathscr{R}[J]$ is contained in the radical of A, which is $\{0\}$. By hypothesis, $B = J \oplus \mathscr{R}[J]$. Then, given $a \in A$, we see that $a = b + c$, where $b \in J$, $c \in \mathscr{R}[J]$. But then $c \in A \cap \mathscr{R}[J] = \{0\}$. Therefore A = J.

Now we present without formal proof an application of Theorem 10 to certain rings of vector-valued functions (Naimark's terminology [8]). Let Ω be a nonempty set considered with the discrete topology. Let $\mathscr{H}$ be a Hilbert space, $\mathscr{K}(\mathscr{H})$ the algebra of all compact operators on $\mathscr{H}$, and when $T \in \mathscr{K}(\mathscr{H})$, let $|T|$ denote the operator norm of T. We define B to be the algebra of all functions $f: \Omega \to \mathscr{K}(\mathscr{H})$ that vanish at ∞ on Ω (that is, to each f and each $\varepsilon > 0$ there corresponds a finite subset $\Gamma_\varepsilon \subset \Omega$ such that $|f(\alpha)| < \varepsilon$ whenever $\alpha \in \Omega \setminus \Gamma_\varepsilon$). B is a B*-algebra in the norm

$$\|f\| = \sup\{|f(\alpha)| \mid \alpha \in \Omega\}.$$

Furthermore, B is an annihilator algebra. Given $\alpha \in \Omega$ and $\phi \in \mathscr{H}$, $\phi \neq 0$, set

$$M(\alpha, \phi) = \{f \in B \mid f(\alpha)\phi = 0\}.$$

The maximal left ideals of B are exactly the left ideals of the form $M(\alpha, \phi)$.

THEOREM 11. *Assume that* A *is a closed subalgebra of* B *such that*

(i) *for each* $\alpha \in \Omega$ *and each* $T \in \mathscr{K}(\mathscr{H})$, *there exists* $f \in A$ *such that* $f(\alpha) = T$, *and*

(ii) *for all* $\alpha, \beta \in \Omega$ *and all* $\phi, \psi \in \mathscr{H}$ *such that either* $\alpha \neq \beta$ *or* ϕ *and* ψ *are linearly independent, there exists* $f \in A$ *such that* $f(\alpha)\phi = 0$ *and* $f(\beta)\psi \neq 0$.

Then A = B.

Using property (i), we can show that $M(\alpha, \phi) \cap A$ is a maximal modular left ideal of A for each choice of $\alpha \in \Omega$ and $\phi \in \mathscr{H}$ ($\phi \neq 0$). It follows that A is semisimple. Property (ii) implies immediately that A separates the maximal left ideals of B. Then the theorem follows from Theorem 10.

Theorem 11 generalizes [8, Theorem 6, p. 348] in the case where the underlying space has the discrete topology.

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