CONVEX HYPERSURFACES

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1. INTRODUCTION

Concerning the relation between the topology and curvature K of a Riemannian manifold, it is known that

(i) an Hadamard manifold (a complete, simply-connected manifold with $K \leq 0$) is diffeomorphic to Euclidean space,

(ii) if a simply-connected, complete manifold is 1/4-pinched (that is, if $1/4 < K \leq 1$), it is homeomorphic to a sphere, and

(iii) a complete open manifold of positive curvature and of dimension at least 5 must be diffeomorphic to Euclidean space.

In this paper, we investigate hypersurfaces that are embedded in an Hadamard manifold or in a 1/4-pinched complete Riemannian manifold and satisfy the semiconvexity condition defined in Section 2. We prove the following two theorems.

THEOREM A. *Let* M^n $(n \neq 4, 5)$ *be a simply-connected,* 1/4-*pinched, complete Riemannian manifold, and let* N^{n-1} *be a simply-connected, semiconvex, compact hypersurface embedded in* M. *Then* N *is homeomorphic to* S^{n-1}.

THEOREM B. *Every semiconvex, compact hypersurface embedded in an Hadamard manifold is diffeomorphic to a sphere.*

The proofs use a modification of an argument due to Hadamard. The restriction on n in Theorem A arises from the application of a theorem in [5, p. 264]. Both theorems generalize the results of F. J. Flaherty [3], [4].

2. CONVEX HYPERSURFACES AND STAR-SHAPED SETS

Let M^n be a Riemannian manifold diffeomorphic either to R^n or to S^n. By a well-known separation theorem, each compact, connected embedded hypersurface N divides M into two components. On the other hand, suppose z is a fixed unit normal vector field on N in M, and let r denote the injectivity radius of N in M [5]; define two subsets of M - N as

$$A = \{\operatorname{Exp} tz\colon 0 < t \leq r\} \quad \text{and} \quad B = \{\operatorname{Exp} t(-z)\colon 0 < t \leq r\}. \tag{1}$$

Both A and B are the images of connected sets under the continuous map Exp. Consequently, both are connected, and $A \cup B = M - N$. However, by the separation theorem, M - N has exactly two components, and therefore A and B must be the components of M - N.

Next, recall the second fundamental form L_z, defined by the equation

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(2) $$\langle L_z(x), y \rangle = \langle \overline{\nabla}_x z, y \rangle ,$$

where x and y are tangent vectors of N and $\overline{\nabla}$ is the covariant differentiation of M. Call an embedded hypersurface N *convex* (respectively, *semiconvex*) if L_z is positive definite (respectively, positive semidefinite) for a unit normal vector field z on N. For a convex or semiconvex hypersurface N, define

(3) $$N^+ = A \cup N, \quad N^- = B \cup N,$$

where A and B are the sets defined in (1). Notice that both N^+ and N^- are manifolds with the boundary N. As usual, for each $m \in M$ for which there exists a unique shortest geodesic from m to N, we denote this geodesic by γ_{mN}. It is clear that γ_{mN} is parallel to -z for $m \in N^+$ and parallel to z for $m \in N^-$. Finally, a set S in R^n is called *star-shaped around* $p \in S$ if S contains the whole segment xp for each x in S. The star-shaped property is said to be *strong* if the open segment xp is contained in the interior of S. Similar terminology applies to any Riemannian manifold, provided we replace the words "line segment" by (unique) "geodesic segment".

Suppose now that N lies within the injectivity region of $p \in N^- - N$. We expect to say something about N by lifting N or N^- into the tangent space of M at p by means of the exponential map. Although convexity is usually destroyed, the property of being star-shaped is always preserved.

LEMMA 1. *Let* N *be a convex hypersurface in* M^n. *If* N *lies within the injectivity region of* $p \in N^- - N$, *then* N^- *is a star-shaped set around* p.

Proof. Corresponding to a tangent vector v at p, let γ_v denote the unique geodesic tangent to v at p. Suppose that $\gamma_v(t_0), \gamma_v(t_1) \in N$ but $\gamma_v(t) \notin N$, where $0 < t_0 < t < t_1 < 1$.

By the connectedness of N^-, we can find a smooth curve $\sigma(s)$ $(0 \leq s \leq 1)$ from $\gamma_v(1)$ to a point $\sigma(1)$ so close to p that the shortest geodesic joining $\sigma(1)$ to p lies within $N^- - N$. Now construct a smooth family of geodesics

$$\gamma^s(t) \quad (0 \leq t \leq 1, \; 0 \leq s \leq 1)$$

by joining p to $\sigma(s)$ via the shortest geodesic $\gamma^s(t)$ $(0 \leq t \leq 1)$. It is clear that there exist s and t_2 such that

$$\gamma^s(t_2) \in N \quad \text{and} \quad \gamma^s(t) \in N^- \quad \text{for all } t.$$

By a variational argument, γ^s must be tangent to N at t_2, and hence γ^s lies locally in $N^+ - N$ near t_2 (since N is convex). This contradiction establishes the lemma.

LEMMA 2. *Let* S *be a set in* R^n *that is star-shaped around* 0 *and has a smooth hypersurface* N *as boundary. Then* N *is diffeomorphic to* S^{n-1}.

Proof. We shall fatten S a little, so that it becomes strongly star-shaped around 0. Let n_x denote the outward unit normal vector field. By the tubular-neighborhood property, for small positive ε, the set

$$N^\varepsilon = \{x + \varepsilon n_x : x \in N\}$$

is a smooth manifold diffeomorphic to N. Moreover, if we construct

$$N^-(\varepsilon) = \{x \in R^n : \text{dist}(x, N^-) \leq \varepsilon\},$$

then N^ε is the boundary of $N^-(\varepsilon)$. We claim that $N^-(\varepsilon)$ and N^ε are nice in the sense that

(i) $N^-(\varepsilon)$ is strongly star-shaped around 0, and

(ii) each radial vector intersects N^ε transversally.

Since $d(x + \varepsilon n_x) = dx + \varepsilon\, dn_x \perp n_x$, the normal of N^ε at $x + \varepsilon n_x$ is parallel to n_x. Hence it suffices to prove that

$$\langle x + \varepsilon n_x, n_x \rangle = \langle x, n_x \rangle + \varepsilon \neq 0.$$

But this is true for sufficiently small ε, because $\langle x, n_x \rangle \geq 0$.

Finally, (i) and (ii) prove the lemma. The diffeomorphism between N^ε and S^{n-1} can be constructed explicitly.

3. PROOF OF THEOREM A

In order to apply Lemma 1, we need a perturbation lemma for the positively curved ambient manifold. Let M^n be a Riemannian manifold all of whose sectional curvatures are strictly positive, and let N^{n-1} be a compact semiconvex hypersurface in M. Fix a unit normal z of N so that all second fundamental forms of N along z are negative semidefinite. For small $\varepsilon > 0$, consider the parallel hypersurface

$$N^\varepsilon = \{\text{Exp}\, \varepsilon z(x) : x \in N\}.$$

LEMMA 3. *For sufficiently small, positive* ε, *the hypersurface* N^ε *is strictly convex.*

Proof. By the compactness of N, we may assume that within the tubular neighborhood of N the sectional curvature k(M) of M lies in $[\delta, \Delta]$ $(0 < \delta \leq \Delta < \infty)$. A first variation argument shows that extending z along the geodesic perpendicular to N will give us a unit normal vector of N^ε. Since the length function from N to N^ε has constant value ε, the second variation (the index) must vanish identically. Moreover, all longitudinal curves are geodesics perpendicular to both N and N^ε, and the second fundamental form L^ε of N^ε has the expression

$$\langle L^\varepsilon(V), V \rangle = \langle V', V \rangle(\varepsilon),$$

where V is an N-Jacobi field along the geodesic perpendicular to N.

It remains to prove that for all such Jacobi fields V, the inequality

$$\langle V', V \rangle(\varepsilon) < 0 \tag{4}$$

holds for sufficiently small $\varepsilon > 0$. Denote the geodesic perpendicular to N by $\gamma: [0, \varepsilon] \to M$. Choose a parallel orthonormal frame E_i along γ such that $E_n = \gamma'$ and $L(E_i) = \lambda_i E_i$ $(1 \leq i < n)$, where $\lambda_i \leq 0$.

Set $V = \sum_{i=1}^{n-1} f_i E_i$ with $\|V(0)\|^2 = \sum_{i=1}^{n-1} (f_i(0))^2 = 1$. Then the relation $V'' = R_{V\gamma',\gamma'}$ implies that

(5) $$f_i'' = \sum_{j=1}^{n-1} f_j k_{ji}, \quad \text{where } k_{ji} = \langle R_{E_j E_n} E_n, E_i \rangle,$$

and the boundary condition $L(V(0)) - V'(0) = 0$ gives the equation

$$\sum_{i=1}^{n-1} (f_i \lambda_i E_i - f_i' E_i)(0) = 0;$$

that is,

(6) $$f_i(0)\lambda_i = f_i'(0) \quad (1 \leq i \leq n-1).$$

Set $g(t) = \sum_{i=1}^{n-1} f_i(t) f_i'(t) = \langle V', V \rangle (t)$. Then use (5) and (6) to obtain the formulas

$$g(0) = (f_1(0))^2 \lambda_1 + \cdots + (f_{n-1}(0))^2 \lambda_{n-1}$$

and

$$g'(0) = \sum_{i=1}^{n-1} (f_i(0))^2 \lambda_i^2 + \sum_{i,j=1}^{n-1} f_i(0) f_j(0) k_{ji}$$

$$= \sum_{i=1}^{n-1} f_i(0)^2 \lambda_i^2 + \langle R_{VE_n} E_n, V \rangle = \sum_{i=1}^{n-1} (f_i(0))^2 \lambda_i^2 - K_{VE_n},$$

where K_{VE_n} is the curvature of the section spanned by E_n and V. Assume $0 < \delta \leq K_{VE_n} \leq \Delta < \infty$. It remains to prove that $g(\varepsilon) < 0$ for a sufficiently small ε. We notice that both $g(0)$ and $g'(0)$ can be considered as continuous functions on the unit tangent bundle U of N, with

$$g'(0) \leq (f_1(0))^2 \lambda_1^2 + \cdots + (f_{n-1}(0))^2 \lambda_{n-1}^2 - \delta.$$

There are two possibilities:

(i) If $(f_1(0))^2 \lambda_1^2 + \cdots + (f_{n-1}(0))^2 \lambda_{n-1}^2 < \delta/2$, then $g'(0) < -\delta/2$.

(ii) If $(f_1(0))^2 \lambda_1^2 + \cdots + (f_{n-1}(0))^2 \lambda_{n-1}^2 \geq \delta/2$, then $g(0) \leq -\delta/2B$, where $-B$ is the lower bound on the eigenvalues of all second fundamental forms of N. Both cases imply that

$$\min \{B g(0), g'(0)\} \leq -\delta/2.$$

Hence we can find an ε such that $\min \{B g(t), g'(t)\} < -\delta/4$ for all t $(0 \leq t \leq \varepsilon)$. Such an ε will prove our claim. Let the number α be defined by the condition

$$B g(\alpha) \leq -\delta/4 \quad \text{and} \quad B g(t) > -\delta/4 \quad \text{for } \alpha < t \leq \varepsilon$$

(if no such number exists, let $\alpha = 0$). Then $g'(t) < -\delta/4$ for $\alpha < t \leq \varepsilon$, and $g(\varepsilon) = g(\alpha) + \int_\alpha^\varepsilon g'(t)\,dt < 0$; this gives (4).

Proof of Theorem A. By Lemma 3, we may assume N to be strictly convex. If we use k(M) and k(N) to denote the sectional curvature of M and N respectively, then the Gauss equation gives the inequalities

$$1/4 < k(M) \leq k(N).$$

Denote by d(N) (by d'(N)) the diameter of N in the metric of N (in the metric of M). Then we have two possibilities:

(i) If $d(N) \geq \pi$, then by a result due to Berger [5, p. 264], N is homeomorphic to S^{n-1}, for $n \neq 4, 5$.

(ii) If $d(N) < \pi$, then $d'(N) \leq d(N) < \pi$. Pick $p \in N^- - N$ so close to N that $d(p, x) < \pi$ for all $x \in N$. By a result due to W. Klingenberg [5, p. 254], the injectivity radius d_p satisfies the inequality $d_p \geq \pi/\sqrt{1} = \pi$. Hence N^- lies completely in the injectivity region of p. By Lemma 1, N^- is star-shaped around p. Now lift to the tangent space at p; this gives us a star-shaped region in R^n with smooth boundary Exp_p^{-1} N. An application of Lemma 2 to the tangent space completes the proof.

4. PROOF OF THEOREM B

Since in an Hadamard manifold we have no perturbation lemma corresponding to Lemma 3, methods analogous to those of Section 3 do not apply here. Let the manifolds N and M have the properties listed in Section 2; in addition, suppose that M is an Hadamard manifold. Fix a unit normal vector field z so that L_z is positive semidefinite, and define the set

$$\perp N^+ = \{tz: 0 \leq t < \infty\}.$$

Also, define the mapping $f: N^+ \to R$ by the equation $f(p) = \mathrm{dist}(p, N)$.

LEMMA 4. $\mathrm{Exp} \mid \perp N^+$ *is a diffeomorphism, and* $f \in C^\infty$.

Proof. By the generalized Rauch comparison theorem [6, Theorem 4.1], $\mathrm{Exp} \mid \perp N^+$ is nonsingular. We prove that $\mathrm{Exp} \mid \perp N^+$ is a one-to-one map. Suppose it is not, then we can find $v \neq w \in \perp N^+$ such that

$$\mathrm{Exp}\, v = \mathrm{Exp}\, w = p.$$

Let

$$\alpha(t) = \mathrm{Exp}\, tv, \quad \beta(t) = \mathrm{Exp}\, tw \qquad (0 \leq t \leq 1).$$

As usual, we use $\Omega(p, N)$ to denote the space of all curves from N to p. Since $\Omega(p, N)$ is connected, we have a homotopy

$$H: [0, 1] \times [0, 1] \to M$$

such that

$$H(t, 0) = \alpha(t) \quad \text{and} \quad H(t, 1) = \beta(t) \qquad \text{for all } t,$$

$$H(0, s) \in N \quad \text{and} \quad H(1, s) = p \qquad \text{for all } s.$$

By lifting to $\perp N^+$, we shall mean the lift through $\mathrm{Exp} \mid \perp N^+$. It is clear that $H \mid [0, 1] \times \{0\}$ can be lifted to a map $\Omega: [0, 1] \times \{0\} \to \perp N^+$, since $\mathrm{Exp} \mid \perp N^+$ is nonsingular. Suppose J_0 is the set of all η such that $H \mid [0, 1] \times [0, \eta]$ can be lifted to $\Omega: [0, 1] \times [0, \eta] \to \perp N^+$. Then, by an argument of [5, p. 199], J_0 is both open and closed. Hence $J_0 = [0, 1]$. However, this is impossible, since under the lifting procedure

$$\Omega(1, s) = v \quad \text{for all } s \qquad \text{and} \qquad \Omega(t, 1) = tw \quad \text{for small } t.$$

The latter equation implies that $\Omega(t, 1) = tw$ for all t, which contradicts the former. This completes the proof that $\mathrm{Exp} \mid \perp N^+$ is a diffeomorphism. Next, by the inverse-function theorem, $g = (\mathrm{Exp} \mid \perp N^+)^{-1} \in C^\infty$ on N^+, and hence $f(x) = \|g(x)\| \in C^\infty$ over N^+.

LEMMA 5. *The function* f *is convex in the sense of Section* 2 *of* [1].

Proof. Let $\alpha: [-1, 1] \to N^+$ be a geodesic in M. Construct the rectangle

$$r(u, v) = \gamma_{\alpha(v)N}(u).$$

By Lemma 4, $r \in C^\infty$. Denote by $\ell(v)$ the length function of $r(\cdot, v)$; then since α is a geodesic,

$$\ell(v)\,\ell''(v) = \int_0^1 \{\|r_{vu}^\perp\|^2 - \langle R_{r_v r_u} r_v, r_u \rangle\}\,du + \langle A, r_u \rangle \Big|_{u=0}^{u=1},$$

where $r_{vu}^\perp$ is the projection of r_{vu} onto the normal space $r_u^\perp$ of r_u and $A(u, v) = r_{vv}(u, r)$, $A(0, v) = 0$. Since $K \leq 0$,

$$\ell(v)\,\ell''(v) = \text{nonnegative} + \langle \overline{\nabla}_{r_v} r_v, r_u \rangle(1) = \text{nonnegative} - \langle L_{r_u}(r_v), r_v \rangle(1) \geq 0.$$

Because $\ell''(0)$ is simply the Hessian of f evaluated at $(\alpha'(0), \alpha'(0))$, the lemma follows immediately.

COROLLARY. N^- *is totally convex; that is,* N^- *contains all geodesic segments joining any two of its points. In particular,* N^- *is star-shaped around each point of* N^-.

Proof of Theorem B. By the preceding corollary, N^- is star-shaped around $p \in N^-$. Since Exp_p is a diffeomorphism, we can lift N^- and N to the tangent space at p, which gives us a star-shaped region in R^n (perhaps only weakly star-shaped) with smooth boundary $\mathrm{Exp}_p^{-1} N$. Apply Lemma 2 to the tangent space to complete the proof.

5. REMARKS

(i) Simple connectedness is essential to Theorem B, as the following counterexample shows: Take some nonspherical closed surface F of constant negative curvature. Let B denote the real line with the usual metric, and pick some strictly convex, positive function f on B such that $f(0) = 1$. Let M be $B \times_f F$, the warped product of B and F [1]. It was proved in [1] that M has negative curvature K and that each vertical fibre $\pi^{-1}(b)$ with $b \neq 0$ is an embedded hypersurface with definite second fundamental form. However, each $\pi^{-1}(b)$ is diffeomorphic to a nonspherical surface F.

(ii) If we add the requirement of simple connectedness to N, we may delete it from M, as follows:

THEOREM C. *Let* K: $\hat{M} \to M$ *be a simply-connected Riemannian covering of* M *such that every semiconvex hypersurface of* $\hat{M}$ *is diffeomorphic (respectively, homeomorphic) to the sphere. Then the same is true for every simply-connected, semiconvex hypersurface of* M.

Proof. Let the hypersurface of M be N. Lift N to a map f: $N \to \hat{M}$, which must be one-to-one, and which is therefore an embedded semiconvex hypersurface of $\hat{M}$. By hypothesis, it is diffeomorphic (respectively, homeomorphic) to a sphere.

(iii) To see that N^- is an n-cell, we need only recall the Schoenflies Theorem (see [2]).

(iv) It might be conjectured that similar results hold for the manifolds mentioned in (iii), Section 1. Lemma 3 shows that we need only consider strictly convex hypersurfaces N.

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