SECTIONING BUNDLES OF HIGH FILTRATION AND IMMERSIONS

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Let M^n be a closed orientable manifold of dimension n, and let ν be its stable normal bundle. M. W. Hirsch [7] has shown that if $k < n$, then M^n immerses in R^{2n-k} if and only if the geometric dimension of ν is at most $n - k$. Classifying ν by a map $\nu: M^n \to BSO(n+1)$, we find that an immersion is equivalent to a lifting

$$\begin{array}{ccc} & & BSO(n-k) \\ & \nearrow & \downarrow \pi_0 \\ M^n & \xrightarrow{\nu} & BSO(n+1). \end{array}$$

If $k < n/2$, then the obstruction theory developed by M. Mahowald [8], [4] and E. Thomas [14], [15] can be applied. This involves trying to compute the obstructions with higher-order cohomology operations, and the method becomes unwieldy for large k, because the construction of operations of order greater than 2 is difficult. In this note we show that if BSO is replaced by its k-connected covering, then the obstructions to lifting any bundle of filtration $k + 1$ can be expressed in terms of higher-order operations that are defined on a generalized cohomology theory $H^*(-; \mathfrak{X})$. The spectrum $\mathfrak{X}$ is simple enough so that these operations can be computed for the normal bundle, and we prove the following result.

THEOREM. *Let* M^n *be a closed orientable manifold of dimension* n, *and let* k *be an integer such that* $2k < n$. *If*

(i) M *is* $(k-2)$*-connected and*

(ii) *the normal bundle* ν *of* M *is trivial over the* k*-skeleton,*

then M^n *immerses in* R^{2n-k} *if and only if* $w_{n-k+1}(\nu) = 0$.

A. Haefliger and M. W. Hirsch [6] have shown that condition (i) gives an immersion in R^{2n-k+1} if and only if $w_{n-k+1}(\nu) = 0$. J. Becker [2] has proved our theorem with a condition slightly weaker than (ii), namely, that ν is fibre-homotopy trivial over the k-skeleton. These results apply only to immersions however, while the techniques used to prove our theorem apply to the problem of sectioning any bundle.

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1. A COHOMOLOGY THEORY

Let k be a positive integer, which will remain fixed throughout the rest of the paper. For $q > k$, let X_q denote the universal example for classes in $H^q(-; Z)$ on which all cohomology operations $\Phi: H^q(-; Z) \to H^{q+i}(-; Z_2)$ vanish for $i \leq k$. Then there exists a map $\phi_q: X_q \to K(Z, q)$, and

(a) X_q is $(q - 1)$-connected, and $x_q = \phi_q^*(\iota_q)$ generates the group $H^q(X_q; Z) = Z$,

(b) $H^i(X_q; Z_2) = 0$ for $q < i \leq q + k$,

$$\text{(c)}\quad \pi_i(X_q) = \begin{cases} \pi_q(S^q) = Z & (i = q), \\ \pi_i(S^q)_2 & (q < i < q + k), \\ 0 & \text{otherwise}. \end{cases}$$

Since the k-invariants for X_q are stable, X_q generates an Ω-spectrum $\mathfrak{X}$, and a cohomology theory is defined by the equation $H^n(B; \mathfrak{X}) = [B, X_n]$ (see [18]).

Primary operations $H^n(-; \mathfrak{X}) \to H^{n+i}(-; Z_2)$ correspond in the usual way to classes in $H^{n+i}(X_n; Z_2)$, such as $Sq^i x_n$ $(k < i \leq n)$. We can obtain universal examples for operations of higher order by constructing a sequence of principal fibrations over X_n. The following result will be necessary for the construction of the operations.

LEMMA 1. *Let* $s: S^{k+1} X_{n-k} \to X_{n+1}$ *be the natural map. Then*

(i) $s^*: H^i(X_{n+1}; Z_2) \to H^i(S^{k+1} X_{n-k}; Z_2)$ *is surjective for* $i \leq 2n + 1$, *and*

(ii) *in dimensions not exceeding* $2n + 2$, Ker s^* *is generated by* $Sq^i x_{n+1}$ $(n - k + 1 \leq i \leq n + 1)$.

Proof. Since the diagram

$$\begin{array}{ccc} H^*(X_{n+1}) & \xrightarrow{\;s^*\;} & H^*(S^{k+1} X_{n-k}) \\ & {\scriptstyle \sigma^{k+1}} \searrow \quad \nearrow {\scriptstyle \approx} & \\ & H^*(X_{n-k}) & \end{array}$$

is commutative, it suffices to prove the corresponding facts for σ^{k+1}.

To prove (i), we must show that $H^i(X_{n-k})$ is stable for $i < 2n - k$. By [17, Corollary 6.3], a class in this range of dimensions is stable if and only if it is primitive. Let $i_q: F_q \to X_q$ be the inclusion of the fibre of ϕ_q, and let m denote the H-map for either F_q or X_q. The only possibility for a nonprimitive class in $H^m(X_q)$ $(m \leq 2q + k)$ would be a class $u \in H^{2q}(X_q)$ such that

$$m^*(u) = u \otimes 1 + x_q \otimes x_q + 1 \otimes u.$$

The last equation implies that

$$m^*(i_q^* u) = i_q^* u \otimes 1 + 1 \otimes i_q^* u,$$

so that $i_q^* u = \sigma(v)$ for some $v \in H^{2q+1}(F_{q+1})$.

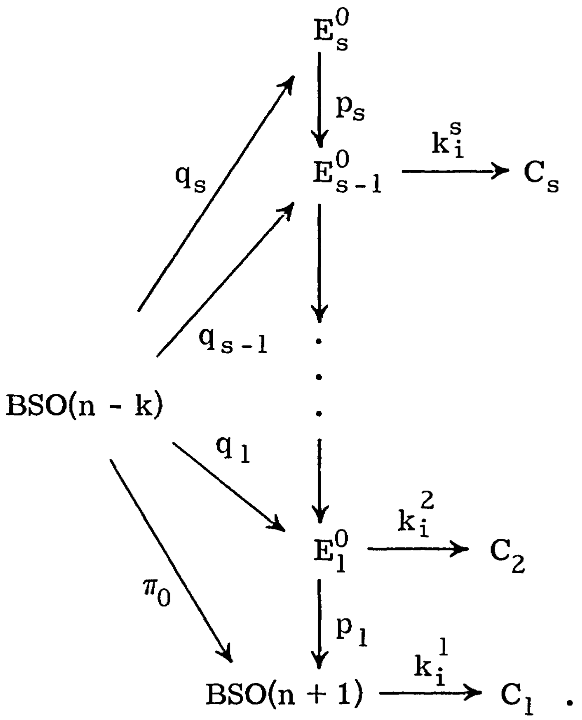
A simple argument using the spectral sequence for ϕ_{q+1} shows that v is transgressive and $\tau(v) = 0$, so that $v = i^*_{q+1}(u')$ for some $u' \in H^{2q+1}(X_{q+1})$. Now $\sigma(u') - u \in \text{Ker } i^*_q$, and looking again at the spectral sequence for ϕ_q, we see that $\text{Ker } i^*_q = \text{Im } \phi^*_q$ in dimension 2q. Since $\text{Im } \phi^*_q$ is stable in dimension 2q, u must be stable, and this completes the proof of (i).

To prove (ii), we use [17, Corollary 6.4], and we find that in dimensions at most $2n + 2$, $\text{Ker } \sigma^{k+1}$ consists of classes u such that $\sigma^{j+1}(u)$ is a product in $H^*(X_{n-j})$ for some $j \leq k$. But $x^2_{n-j} = Sq^{n-j} x_{n-j}$ is the only product in this range of dimensions; therefore $u = Sq^{n-j} x_{n+1} + v$, where $v \in \text{Ker } \sigma^{j+1}$. The same reasoning shows that $\sigma^{j+1} \mid H^{2n-j+1}(X_{n+1})$ is injective; hence $u = Sq^{n-j} x_{n+1}$, as we claimed.

2. OBSTRUCTIONS

A. H. Copeland, Jr., and M. Mahowald [3] have shown that if the integral and 2-primary obstructions to factoring a map through π_0: $BSO(n - k) \to BSO(n + 1)$ vanish, then the p-primary obstructions vanish, for odd primes p. We therefore construct a resolution of π_0 over the mod-2 Steenrod algebra A, which looks like the diagram

(1)



Each C_r is a product of Eilenberg-Maclane spaces $K(Z_2, q)$, and the classes $k^r_i \in H^*(E_{r-1})$ form a minimal set of generators for $\text{Ker } q^*_{r-1}$ over the twisted tensor product $A(BSO(n + 1))$ (see [11]). The fibre of π_0 is $V = SO(n + 1)/SO(n - k)$, and after a finite number s of stages, the 2-primary homotopy of V has been killed through dimension n - 1. (If n - k is even, so that $\pi_{n-k}(V) = Z$, then an integral k-invariant $\delta^* w_{n-k}$ must be used to kill this group.) If B is a complex of dimension n, then by the result of [3] a map $B \to BSO(n + 1)$ lifts to $BSO(n - k)$ if and only if it lifts to E_s.

Let $MSO(n + 1)$ denote the Thom complex of the universal bundle over $BSO(n + 1)$, and let $U \in H^{n+1}(MSO(n + 1); Z)$ denote the Thom class. If $B \to BSO(n + 1)$ is a map, let MB and U_B denote the Thom complex and Thom class of the induced bundle over B. (In the following, there will be only one such map, and it is deleted from the notation.) In particular, $M(BSO(n-k)) = S^{k+1}MSO(n-k)$, and π_0 induces a map $M_{\pi_0}: S^{k+1}MSO(n-k) \to MSO(N+1)$.

A straightforward extension of the results of [10] and [15] (see [5; Lemma 2.11]) gives a commutative diagram

(2)
$$\begin{array}{ccccc}
ME_s^0 & \xrightarrow{f_s} & \tilde{E}_s & & \\
\downarrow & & \downarrow \tilde{p}_s & & \\
\vdots & & \vdots & & \\
\downarrow & & \downarrow & & \\
ME_1^0 & \xrightarrow{f_1} & \tilde{E}_1 & \xrightarrow{\tilde{k}_i^2} & D_2 \\
\downarrow & & \downarrow \tilde{p}_1 & & \\
MSO(n+1) & = & MSO(n+1) & \xrightarrow{\tilde{k}_i^1} & D_1
\end{array},$$

where D_r is a product of Eilenberg-Maclane spaces, $\tilde{p}_r$ is the principal fibration induced by the classes $\tilde{k}_i^r$, and

$$f_{r-1}^*(\tilde{k}_i^r) = U_{E_{r-1}} \cdot k_i^r.$$

Let B_q denote the k-connected covering of $BSO(q)$ for $q > k$. The natural map $B_{n+1} \to BSO(n+1)$ induces a map $\pi: B_{n-k} \to B_{n+1}$ with fibre V; we want to construct a resolution of π. For this, π^* must be surjective in dimensions not exceeding $n+1$, and it follows from the Serre exact sequence that this is so if and only if $w_i \neq 0$ in $H^*(B_{n+1})$ for $i = n-k+1, \cdots, n+1$. Using the results of R. E. Stong [12], one can show that if B is the k-connected covering of BSO, then $w_i \neq 0$ in $H^*(B)$, for each i greater than some number depending on k. (A rough upper bound for this number is $2^{2\phi(0,k+1)}$, where $\phi(0, k+1)$ is defined as in [12].)

If π^* is surjective, there is a resolution

$$\begin{array}{ccccccccc}
 & & C_s & & & & C_2 & & C_1 \\
 & & \uparrow k_i^s & & & & \uparrow k_i^2 & & \uparrow k_i^1 \\
E_s & \longrightarrow & E_{s-1} & \longrightarrow & \cdots & \longrightarrow & E_1 & \longrightarrow & B_{n+1}
\end{array}$$

of π, induced from the resolution of π_0 by the natural map $B_{n+1} \to BSO(n+1)$. We also obtain a diagram similar to diagram (2) with $MSO(n+1)$ replaced by MB_{n+1}. Since B_{n+1} is k-connected, so that the Thom isomorphism takes a set of A-generators for $\operatorname{Ker} q_r^*$ to a set of A-generators for $\operatorname{Ker} M_q^*$, one can easily show that $f_r^*: H^i(\tilde{E}_r; Z) \approx H^i(ME_r; Z)$ for $i \leq 2n+1$. Thus

$$ME_s \longrightarrow \cdots \longrightarrow ME_1 \longrightarrow MB_{n+1}$$

looks like a resolution of M_π through dimension $2n+1$.

Let $\tilde{U} \in H^{n+1}(MB_{n+1}; \mathfrak{X})$ be the unique lifting of the Thom class

$$U \in H^{n+1}(MB_{n+1}; Z).$$

(By obstruction theory, this lifting is unique.) There is then a commutative diagram

$$\begin{array}{ccc}
G & \xrightarrow{\ \eta\ } & F \\
\downarrow & & \downarrow \\
MB_{n-k} & \xrightarrow{\ h\ } & S^{k+1}X_{n-k} \\
\downarrow{\scriptstyle M_\pi} & & \downarrow{\scriptstyle s} \\
MB_{n+1} & \xrightarrow{\ \tilde{U}\ } & X_{n+1}
\end{array},$$

where F and G are the fibres of s and M_π, respectively.

LEMMA 2. $\eta^*: H^i(F; Z_2) \to H^i(G; Z_2)$ *is an isomorphism for* $i \leq 2n + 1$.

Proof. $M_{\pi *}$ and s^* are surjective through dimension $2n + 1$ (see Lemma 1); hence the corresponding transgression operators are injective. By the Serre exact sequence, it suffices to show that $\tilde{U}^*$: Ker $s^* \approx$ Ker $M_{\pi *}$ through dimension $2n + 2$. Since

$$\tilde{U}^*(Sq^i x_{n+1}) = U_{B_{n+1}} \cdot w_i$$

and Ker $M_{\pi *}$ is generated by $U_{B_{n+1}} \cdot w_i$ $(n - k + 1 \leq i \leq n + 1)$, this assertion follows from part (ii) of Lemma 1.

A resolution for s induces a resolution for M_π, by Lemma 2. The k-invariants for s represent higher-order operations $\phi_i^r: H^*(-; \mathfrak{X}) \to H^*(-; Z_2)$, and there is a diagram

$$\begin{array}{ccccc}
ME_s & \xrightarrow{\ f_s\ } & Y_s & & \\
\downarrow & & \downarrow & & \\
\vdots & & \vdots & & \\
\downarrow & & \downarrow & & \\
ME_1 & \xrightarrow{\ f_1\ } & Y_1 & \xrightarrow{\ \phi_i^2\ } & D_2 \\
\downarrow & & \downarrow & & \\
MB_{n+1} & \xrightarrow{\ \tilde{U}\ } & X_{n+1} & \xrightarrow{\ \phi_i^1\ } & D_1
\end{array}$$

such that $f_{r-1}^* \phi_i^r = U_{E_{r-1}} \cdot k_i^r$. We express this by putting

$$(U_{E_{r-1}} \cdot k_i^r) \in (\Phi_i^r)(\tilde{U}_{E_{r-1}}) ,$$

where (Φ_i^r) is thought of as a multi-valued operation. If B is a complex and $\xi: B \to B_{n+1}$ is a map that lifts to E_{r-1}, then putting

$$k_i^r(\xi) = \{\bar{\xi}^* k_i^r \mid p_1 \cdots p_{r-1} \bar{\xi} = \xi\} ,$$

we see that $(U_B \cdot k_i^r(\xi)) \in (\Phi_i^r)(\tilde{U}_B)$. Moreover, it follows from Lemma 1 that the indeterminacy is the same on both sides, so that the problem of evaluating the $k_i^r(\xi)$ is reduced to computing the operation (ϕ_i^r).

3. THE GENERAL CASE

If $F \xrightarrow{j} E \xrightarrow{\pi} B$ is a fibration with q-connected fibre F, then in order to construct a resolution for π through dimension $n \leq 2q$ ([4], [8]), we require that

$H^*(F)$ is transgressive through dimension n and

$\pi^*: H^*(B) \to H^*(E)$ is surjective through dimension n.

If the first condition is satisfied and $u: E \to K = \times_i K(Z_2, r_i)$ represents a set of generators for the A-submodule generated by Coker π^*, it is easily verified that the map $\hat{\pi} = (\pi, u): E \to B \times K$ with fibre $\hat{F}$ satisfies both conditions. Since the diagram

$$\begin{array}{ccc} E & = & E \\ \downarrow{\scriptstyle\hat{\pi}} & & \downarrow{\scriptstyle\pi} \\ B \times K & \longrightarrow & B \end{array}$$

is commutative, the lifting problem for π is equivalent to that for $\hat{\pi}$.

If $\pi^*: H^*(B_{n+1}) \to H^*(B_{n-k})$ is not surjective, then certain Stiefel-Whitney classes w_{r_i+1} are zero in $H^*(B_{n+1})$, giving rise to classes $u_{r_i} \in H^{r_i}(B_{n-k})$ such that $j^*(u_{r_i}) = a_{r_i}$, where a_{r_i} denotes the generator of $H^{r_i}(V)$. The next lemma shows that if r_i is even, the class u_{r_i} is integral.

LEMMA 3. *Let* $f: B \to BSO$ *be a map such that* $f^*(w_{2r+1}) = 0$. *Then* $f^*(\delta w_{2r}) = 0$.

Proof. Let $p: Q \to BSO$ be the principal fibration induced by w_{2r+1}, giving a diagram

$$\begin{array}{ccccc} K(Z_2, 2r) & \longrightarrow & Q & & \\ & & \downarrow{\scriptstyle p} & & \\ B & \xrightarrow{f} & BSO & \xrightarrow{w_{2r+1}} & K(Z_2, 2r+1) . \end{array}$$

Since f lifts to Q, it suffices to show that $p^* \delta w_{2r} = 0$. Now

$$H^{2r}(K(Z_2\,, 2r);\, Z) \approx Z_2\,,$$

and if ι denotes the generator, then $\rho_2\, \tau(\iota) = w_{2r+1} = \rho_2\, \delta w_{2r}$. Thus

$$\tau(\iota) = \delta w_{2r} + 2u$$

for some integral class u. But u is a torsion class ($4u = 0$), and since all torsion in BSO has order 2, we see that $2u = 0$. Therefore $\delta w_{2r} = \tau(\iota)$, and $p^* \delta w_{2r} = 0$.

Let $K = \times_i K(J_i\,, r_i)$, where J_i is Z if r_i is even and Z_2 if r_i is odd, and let $u\colon B_{n-k} \to K$ be a map representing the u_{r_i}. Put $\hat\pi = (\pi, u)\colon B_{n-k} \to B_{n+1} \times K$, and let $\hat V \to V$ be the principal fibration induced by the classes $a_{r_i} \in H^{r_i}(V; J_i)$. There is then a commutative diagram

$$(3)\qquad \begin{array}{ccccc}
\hat V & \longrightarrow & V & \xrightarrow{\;a_{r_i}\;} & K \\
\downarrow & & \downarrow & & \downarrow \\
B_{n-k} & = & B_{n-k} & \longrightarrow & P \\
\downarrow{\scriptstyle \hat\pi} & & \downarrow{\scriptstyle \pi} & & \downarrow \\
B_{n+1} \times K & \xrightarrow{\;p\;} & B_{n+1} & \xrightarrow{\;0\;} & \Omega^{-1}K\,.
\end{array}$$

By a lemma of Thomas [16, Appendix], $\hat V$ is the fibre of $\hat\pi$.

Next we alter $F \to S^{k+1} X_{n-k} \xrightarrow{s} X_{n+1}$ in a similar fashion. For the integers r_i occurring above, let $b_{r_i} \in H^*(F)$ be the unique class transgressing to $Sq^{r_i+1} x_{n+1}$ (if r_i is even, we can replace Sq^{r_i+1} by δSq^{r_i}, making b_{r_i} integral). As above, we obtain a commutative diagram

$$(4)\qquad \begin{array}{ccccc}
\hat F & \longrightarrow & F & \xrightarrow{\;b_{r_i}\;} & \Omega^{-n-1}K \\
\downarrow & & \downarrow & & \downarrow \\
S^{k+1} X_{n-k} & = & S^{k+1} X_{n-k} & \text{———} & P \\
\downarrow{\scriptstyle \hat s} & & \downarrow{\scriptstyle s} & & \downarrow \\
X_{n+1} & \xrightarrow{\;\rho\;} & X_{n+1} & \xrightarrow{\;Sq^{r_i+1}\, x_{n+1}\;} & \Omega^{-n-2}K\,,
\end{array}$$

and the lemma of Thomas shows that $\hat F$ is the fibre of $\hat s$.

Since $w_i = 0$ in $H^*(B_{n+1})$ and $Sq^i x_{n+1} = 0$ for $i \le k$, we see that

$$Sq^i Sq^j x_{n+1} = \binom{j-1}{i} Sq^{i+j} x_{n+1} \quad\text{and}\quad Sq^i w_j = \binom{j-1}{i} w_{i+j} \qquad \text{for } i \le k\,.$$

Comparing diagrams 3 and 4, we conclude that there exist operations $\alpha_{ij} \in A$ such that

(i) in dimensions not exceeding $n+1$, Ker $\hat{\pi}^*$ is generated over A by classes of the form

$$w_i \otimes 1 \quad (i \neq r_j,\ n-k+1 \leq i \leq n+1) \qquad \text{and} \qquad b_i \otimes 1 + \sum_j 1 \otimes \alpha_{ij}\, \iota_{r_j},$$

(ii) in dimensions not exceeding $2n+2$, Ker $\hat{s}^*$ is generated over A by classes of the form $Sq^i \hat{x}_{n+1}$ $(i \neq r_j,\ n-k+1 \leq i \leq n+1)$ and classes ϕ_i that restrict to $\sum_j \alpha_{ij}\, \iota_{r_j+n+1}$ on the fibre of ρ.

We use this information to prove the following lemma.

LEMMA 4. *There exists a commutative diagram*

$$\begin{array}{ccc} \hat{G} & \xrightarrow{\ \hat{\eta}\ } & \hat{F} \\ \downarrow & & \downarrow \\ MB_{n-k} & \xrightarrow{\ h\ } & S^{k+1}X_{n-k} \\ \downarrow{\scriptstyle M_{\hat{\pi}}} & & \downarrow{\scriptstyle \hat{s}} \\ M(B_{n+1} \times K) & \xrightarrow{\ \hat{U}\ } & \hat{X}_{n+1}, \end{array}$$

where $\hat{G}$ *is the fibre of* $M_{\hat{\pi}}$. *Furthermore,* $\hat{\eta}^*: H^i(\hat{F}; Z_2) \approx H^i(\hat{G}; Z_2)$ *for* $i \leq 2n+1$.

Proof. We construct a map $\hat{U}$ such that $\hat{s}h = \hat{U}M_{\hat{\pi}}$ and $\hat{U}^*$: Ker $\hat{s}^* \approx$ Ker $M_{\hat{\pi}}^*$ through dimension $2n+2$. Since $M_{\hat{\pi}}^*$ and $\hat{s}^*$ are surjective through dimension $2n+1$, an argument similar to that used in the proof of Lemma 2 shows that $\hat{\eta}^*$ is an isomorphism through dimension $2n+1$.

Consider the diagram

$$(5) \qquad \begin{array}{ccccc} MB_{n-k} & = & MB_{n-k} & \xrightarrow{\ h\ } & S^{k+1}X_{n-k} \\ \downarrow{\scriptstyle M_{\hat{\pi}}} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \hat{s}} \\ M(B_{n+1} \times K) & \xrightarrow{\ g\ } & MB_{n+1} \times \Omega^{-n-1}K & \xrightarrow{\ f\ } & \hat{X}_{n+1} \\ \downarrow{\scriptstyle M_p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \rho} \\ MB_{n+1} & = & MB_{n+1} & \xrightarrow{\ \tilde{U}\ } & X_{n+1} \end{array},$$

where $g = (M_p, U_{B_{n+1} \times K} \cdot (1 \otimes \iota_{r_i}))$ and $r = (M_\pi, U_{B_{n-k}} \cdot u_{r_i})$. Here g is the projection, and f is some map making the lower right-hand square commutative. Since $\rho\hat{s}h = \rho f r$ and r^* is surjective through dimension $2n+1$, we can alter f, if necessary, by a map $MB_{n+1} \times \Omega^{-n-1}K \to \Omega^{-n-1}K$ (the fibre of ρ) to make the whole diagram commutative. The composition fg is then the desired map $\hat{U}$.

Since the diagram

$$\begin{array}{ccc} MB_{n+1} & \xrightarrow{\tilde{U}} & X_{n+1} \\ \downarrow 0 & & \downarrow Sq^{r_i+1}x_{n+1} \\ \Omega^{-n-2}K & \xrightarrow{=} & \Omega^{-n-2}K \end{array}$$

commutes, f induces the identity map on fibres. Using the remarks preceding the statement of Lemma 4, together with the definition of the map g, we conclude that

$$\hat{U}^* Sq^i \hat{x}_{n+1} = U_{B_{n+1}\times K}(w_i \otimes 1) \quad \text{and} \quad \hat{U}^* \phi_i = U_{B_{n+1}\times K} \cdot (b_i \otimes 1 + \sum_j 1 \otimes \alpha_{ij}\, \iota_{r_j})$$

for some b_i. Since $B_{n+1} \times K$ is k-connected, there is no "twisting," and these classes generate $\operatorname{Ker} M_{\hat{\pi}}{}^*$ over A. Thus $\hat{U}$ has the required properties, and the lemma is proved.

It is not hard to show that $\hat{\pi}$ has a finite resolution if some integral k-invariants are used at the first two stages. This is unnecessary, however, since π_0: BSO(n - k) → BSO(n + 1) has a finite resolution, say of height s, and the first s stages of any resolution for $\hat{\pi}$ fit into a commutative diagram

$$\begin{array}{ccccccc} E_s^0 & \longrightarrow & \cdots & \longrightarrow & E_1^0 & \longrightarrow & BSO(n+1) \\ \uparrow & & & & \uparrow & & \uparrow \\ E_s & \longrightarrow & \cdots & \longrightarrow & E_1 & \longrightarrow & B_{n+1} \times K \end{array}$$

If ξ: B → BSO(n + 1) represents a bundle of filtration k + 1, ξ lifts to $\hat{\xi}$: B → $B_{n+1} \times K$, and a lifting of $\hat{\xi}$ to E_s gives a lifting of ξ to E_s^0.

4. PROOF OF THE THEOREM

As in Section 1, the space $\hat{X}_{n+1}$ generates an Ω-spectrum $\hat{\mathfrak{X}}$ and a cohomology theory $H^*(-; \hat{\mathfrak{X}})$. As we indicated in the remarks at the end of Section 2, a resolution of $\hat{s}$ gives higher-order operations Φ_i^r defined on $H^*(-; \hat{\mathfrak{X}})$, and if k_i^r denotes a suitable choice of k-invariants for $\hat{\pi}$, we obtain the relation

$$(U_{E_{r-1}} \cdot k_i^r) \in (\Phi_i^r)(\hat{U}_{E_{r-1}}) .$$

Let M^n be a manifold satisfying the hypotheses of the theorem. The normal bundle lifts to a map ν: $M^n \to B_{n+1} \times K$; hence, if $M(\nu)$ is the Thom complex, then

$$(U_M \cdot k_i^r(\nu)) = (\Phi_i^r)(\hat{U}_M) .$$

By construction, the indeterminacies are the same, and ν lifts to B_{n-k} if and only if

$$(0, \cdots, 0) \in (\Phi_i^r)(\hat{U}_M) \qquad (r = 1, \cdots, s - 1) .$$

We claim that if $w_{n-k+1}(\nu) = 0$ and M^n is (k - 2)-connected, then the only remaining obstructions are those in the top dimension. We must verify that in passing from π to $\hat{\pi}$, no new k-invariants are introduced in dimensions not exceeding n - k + 1, and this follows from a simple argument using Lemma 3 (for example, if n - k + 1 is odd and $w_{n-k+1} = 0$ in B_{n+1}, we have taken the fundamental class ι_{n-k} of K to be integral, so that no k-invariant of the form $b \otimes 1 + 1 \otimes Sq^1 \iota_{n-k}$ can occur).

The nonzero class in $H^{2n+1}(M(\nu); Z_2)$ is spherical, and by construction of $\hat{\mathfrak{X}}$, $H^{n+1}(S^{2n+1}; \hat{\mathfrak{X}}) = 0$. Therefore each Φ_i^r that lands in the top dimension is zero, and all obstructions vanish. This completes the proof of the theorem.

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