HELSON SETS IN COMPACT AND LOCALLY COMPACT GROUPS

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We continue our investigation (begun in [1] and [4]) of the measure space $M_0(G)$, where G denotes an infinite, nondiscrete, locally compact group, not necessarily abelian. In the present paper, we show that each measure in $M_0(G)$ is continuous. We further show that if G is compact or metrizable, then a Helson set cannot support a nonzero measure in $M_0(G)$ (a *Helson set* is a compact set P in G such that every continuous function on P can be extended to a function in the Fourier algebra A(G) of the group G).

Let G denote an infinite, nondiscrete, locally compact group (not necessarily abelian) with left-invariant Haar measure m_G, and let M(G) denote the space of finite regular Borel measures on G. We use the notation and machinery developed by P. Eymard [5] as well as that in [2]. Let Σ denote the equivalence classes of the continuous unitary representations on G, and for $\pi \in \Sigma$, let $\mathscr{H}_\pi$ denote the representation space. For $\mu \in M(G)$, we define the function $\hat{\mu}$ on Σ by

$$\pi \mapsto \hat{\mu}_\pi = \int_G \pi(x)\, d\mu(x)\,.$$

For $\mathscr{S} \subset \Sigma$, let

$$\|\mu\|_{\mathscr{S}} = \sup\{\|\hat{\mu}_\pi\|_\infty : \pi \in \mathscr{S}\}\,,$$

where $\|\hat{\mu}_\pi\|_\infty$ denotes the operator norm on $\mathscr{H}_\pi$. We define $C^*(G)$ to be the completion of $L^1(G)$ in $\|\cdot\|_\Sigma$ (see [5, p. 187]). Let $\{\rho\}$ denote the subset of Σ containing just the left-regular representation of G on $L^2(G)$. Let $C^*(G)$ denote the completion of $L^1(G)$ in $\|\cdot\|_\rho$ (see [5, p. 187]). If G is abelian or compact, then $C^*(G) = C^*_\rho(G)$.

If $\mu \in M(G)$, we let $\rho(\mu)$ denote the bounded operator defined on $L^2(G)$ by $h \mapsto \mu * h$ $(h \in L^2(G))$ with operator norm $\|\rho(\mu)\|_\rho$. Let $\mathscr{B}(L^2(G))$ denote the bounded operators on $L^2(G)$. Then $C^*_\rho(G)$ can be identified with the closure in $\mathscr{B}(L^2(G))$ of the set $\rho(L^1(G)) = \{\rho(f): f \in L^1(G)\}$. If G is abelian, then $C^*_\rho(G)$ is isomorphic to the space $C_0(\hat{G})$ of continuous functions on the dual group $\hat{G}$ that vanish at infinity; and if G is compact, then $C^*_\rho(G) \cong \mathscr{C}_0(\hat{G})$ (see [1]).

Let VN(G) denote the von Neumann subalgebra of $\mathscr{B}(L^2(G))$ generated by the left translation operators (see [5, p. 210]). If $\mu \in M(G)$, then $\rho(\mu) \in VN(G)$. Furthermore, we have the inclusion $C^*_\rho(G) \subset VN(G)$. If G is abelian, then $VN(G) \cong L^\infty(\hat{G})$; and if G is compact, then $VN(G) \cong \mathscr{L}^\infty(\hat{G})$ (see [1]).

Let B(G) denote the linear subspace of $C^B(G)$ (the continuous bounded functions on G) spanned by the continuous, positive-definite functions. Then B(G) can be

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identified with the dual space of $C^*(G)$ (see [5, p. 192]). For $f \in B(G)$, let $\|f\|_B$ denote the norm of f as a linear functional on $C^*(G)$. Now let A(G) be the closed subalgebra of B(G) generated by the continuous, positive-definite functions with compact support (see [5, p. 208]). If G is abelian, then $A(G) \cong L^1(\hat{G})$; and if G is compact, then $A(G) \cong \mathscr{L}^1(\hat{G})$ (see [1]).

The reader familiar with the abelian or compact case will not be surprised to find that the dual of A(G) is VN(G); that is, $A(G)^* \cong VN(G)$ (see [5, p. 210]). Also, A(G) is a VN(G)-module; that is, for $T \in VN(G)$ and $f \in A(G)$, we define $T * f \in A(G)$ by $\langle T * f, S \rangle = \langle f, \check{T}S \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the pairing of A(G) with its dual space VN(G), and where $\check{T}$ is given by $\langle g, \check{T} \rangle = \langle \check{g}, T \rangle$ (here $\check{g}$ denotes the element of A(G) defined by $g(x) = g(x^{-1})$; see [5, p. 212]). If G is abelian, then $L^1(\hat{G})$ is an $L^\infty(\hat{G})$-module by pointwise multiplication, and if G is compact, then $\mathscr{L}^1(\hat{G})$ is an $\mathscr{L}^\infty(\hat{G})$-module by coordinatewise multiplication. If $\mu \in M(G)$ and $f \in A(G)$, then $\rho(\mu) * f$ is precisely $\mu * f$ [5, p. 215]. The basic inequality that we shall need is the relation $\|T * f\|_A \leq \|T\|_{VN} \|f\|_A$ $(T \in VN(G),\ f \in A(G))$ (see [5, p. 213]).

Let $B_\rho(G)$ denote the functions $f \in B(G)$ for which

$$\sup \left\{ \left| \int_G f(x)\, g(x)\, dm_G(x) \right| : g \in L^1(G),\ \|g\|_\rho \leq 1 \right\} < \infty .$$

Then $B_\rho(G)$ can be identified with the dual space of $C_\rho^*(G)$ (see [5, p. 192]).

In our paper [1], we introduced the notation $M_0(G) = \{\mu \in M(G) : \rho(\mu) \in C_\rho^*(G)\}$. This notation differs by a dash from that of one of our other papers [4]. For measures supported on compact sets, the notational differences disappear (see Proposition 3). We have chosen to define the larger space to prove a slightly stronger result. In particular,

$$L^1\overline{(G)}^\rho \supset L^1\overline{(G)}^\Sigma \supset L^1(G), \qquad \text{since } \|\mu\|_\rho \leq \|\mu\|_\Sigma \leq \|\mu\| \quad (\mu \in M(G)).$$

THEOREM 1. *Let* $\mu \in M_0(G)$. *Then* μ *is continuous.*

Proof. Define the map $E: M(G) \to C$ by $E(\mu) = \mu(\{e\})$ $(\mu \in M(G))$. We begin by showing that E is continuous on M(G) with the norm $\|\cdot\|_\rho$. Let $\{\alpha\}$ be a neighborhood basis of e in G. Let $\{f_\alpha\}$ be a collection of functions from A(G) with the following properties: $f_\alpha(e) = 1$, $\|f_\alpha\|_A = 1$, f_α is positive-definite, and support $(f_\alpha) \subset \alpha$. Now

$$|E(\mu)| = \lim_\alpha \left| \int_G f_\alpha\, d\mu \right| = \lim_\alpha |(\mu * f_\alpha)(e)| \leq \|\mu * f_\alpha\|_A \leq \|\rho(\mu)\|_{VN} \|f_\alpha\|_A$$

$$= \|\rho(\mu)\|_{VN} = \|\mu\|_\rho .$$

Since we can extend E to $\rho\overline{(M(G))}^{VN}$ (closure in VN(G)), it is easy to see that $E(\mu * \mu^*) = \sum_{x \in G} |\mu(\{x\})|^2$, and this implies that $E = 0$ on $L^1(G)$.

Let $\mu \in M_0(G)$. Then $\mu * \mu^* \in C_\rho^*(G)$, since $C_\rho^*(G)$ is a $*$-algebra. Since $E = 0$ on $L^1(G)$, $E = 0$ on $L^1\overline{(G)}^{VN} = C_\rho^*(G)$. Thus $E(\mu * \mu^*)$, which is $\sum_{x \in G} |\mu(\{x\})|^2$, *has the value zero. Thus* μ is continuous. ∎

COROLLARY 2. *If* $\mu \in M(G)$ *and* $\rho(\mu)$ *is unitary, then* $\sum_{x \in G} |\mu(\{x\})|^2 = 1$.

Proof. Observe that $E(\mu * \mu^*) = E(\delta_e) = 1$. ■

Let P be a compact subset of G. We denote by $M_0(P)$ the space $M(P) \cap M_0(G)$, and by $M_{0\Sigma}(P)$ the space

$$\{\mu \in M(P): \mu \in \overline{L^1(G)}^{\Sigma} \cong C^*(G)\}.$$

We now show that the spaces $M_0(P)$ and $M_{0\Sigma}(P)$ coincide.

PROPOSITION 3. *Let* P *be a compact subset of* G. *Then* $M_0(P) = M_{0\Sigma}(P)$.

Proof. The inclusion $M_{0\Sigma}(P) \subset M_0(P)$ is obvious. Our results in [4] show that $M_0(P) \subset \overline{L^1(U)}^{\rho}$, where U is some relatively compact neighborhood of P. It remains to show that the topologies on $L^1(U)$ from the norms $\|\cdot\|_\rho$ and $\|\cdot\|_\Sigma$ are equivalent. This follows from the relation $A(G) \mid U = B(G) \mid U$. ■

Definition. Let $P \subset G$ be a compact subset of G such that $A(G) \mid P = C(P)$ (equivalently, for $\mu \in M(P)$ suppose $\|\mu\|$ is equivalent to $\|\mu\|_\rho$ or $\|\mu\|_\Sigma$). We say then that P is a *Helson set.* Note that this is the same as saying that $B(G) \mid P = C(P)$.

We shall show (under the condition that G is compact or metrizable) that no nonzero measure supported in a Helson set can be in $M_0(G)$.

THEOREM 4. *If* P *is a Helson set in a compact group* G *and* $\mu \in M_0(P)$, *then* $\mu = 0$.

Proof. As expected, the proof is modelled on the abelian analogue due to H. Helson (see [7, p. 119]).

For a bounded Borel function ϕ on P, we let T_ϕ be defined on $M_0(P)$ by the relation

$$T_\phi(\mu) = \int_P \phi\, d\mu \qquad (\mu \in M_0(P)).$$

Now T_ϕ is a continuous linear functional on $M_0(P)$. Since $M_0(P)$ can be identified with a closed subspace of $\mathscr{C}_0(\hat{G})$ *via* the Fourier transform $\mathscr{F}$, we can extend T_ϕ to $\mathscr{C}_0(\hat{G})$. Thus there exists a $\psi \in \mathscr{L}^1(\hat{G}) \cong \mathscr{C}_0(\hat{G})^*$ (see [2, Section 8.3.9]) such that $T_\phi(\mu) = \mathrm{Tr}(\hat{\mu}\psi)$ for $\mu \in M_0(P)$ (Tr denotes the trace). Since the Fourier algebra A(G) of G is isomorphic to $\mathscr{L}^1(\hat{G})$ *via* $\mathscr{F}$, there exists an $f \in A(G) \subset C(G)$ with

$$\int_G \phi\, d\mu = \mathrm{Tr}(\hat{\mu}\psi) = \int_P f\, d\mu \qquad (\mu \in M_0(P)).$$

We now use the fact that $M_0(P)$ is a band [1]. This implies that if $\mu \in M_0(P)$, then so is $g\,d\mu$ $(g \in C(G))$. Hence $\int_P \phi g\, d\mu = \int_P fg\, d\mu$ $(g \in C(G))$. It follows that $\phi\, d\mu = f\, d\mu$.

Let $\mu \in M_0(P)$, and suppose by way of contradiction that $\mu \neq 0$. By Theorem 1, μ is continuous, and thus the support S of μ is a nonempty, perfect subset of P. We shall show that S is not extremally disconnected by proving that under our hypotheses G is metrizable.

Let $\mathscr{H}$ denote the normal subhypergroup in $\hat{G}$ generated by $\{\alpha \in \hat{G}: \hat{\mu}_\alpha \neq 0\}$, and let $H = \mathscr{H}^\perp$ be its annihilator in G; that is, let

$$H = \{x \in G: T_\alpha(x) = I_{n_\alpha} \text{ if } \alpha \in \mathscr{H}\}$$

(see [6]). Now $\mathscr{H}^\perp$ is a closed (hence compact) normal subgroup of G, and $\mathscr{H}^{\perp\perp} = \mathscr{H}$ (where $\mathscr{H}^{\perp\perp} = \{\alpha \in \hat{G}: T_\alpha(x) = I_{n_\alpha} \text{ for all } x \in \mathscr{H}^\perp\}$).

We now show that H is a finite subgroup of G. We need the fact that if H is a Helson set (so that A(H) = C(H)), then H is finite. Several proofs of this are known. For example, observe that A(H) is always weakly sequentially complete [3] but that C(H) is weakly sequentially complete only if H is finite.

Let m_H be the Haar measure on H. Then $\mu = m_H * \mu$, and $\mu(E) = \mu(xE)$ for each Borel set E and each $x \in H$. It follows that S is a union of cosets of H. This implies that H is a Helson set, and therefore H is finite.

Now $(G/H)\hat{} = \mathscr{H}$ [6, p. 784], and this set is countable. Thus G/H is metrizable (as is H). Thus G is metrizable (by the Kakutani-Birkhoff characterization of metrizable groups).

Now we can assert the existence of a point $p \in G$ that is in the closure of each of two disjoint open subsets of S, say V_1 and V_2. Finally, let χ_1 be the characteristic function of V_1; we then have the required contradiction of $\chi_1\, d\mu = f\, d\mu$ (for some $f \in C(G)$). ■

Observe that every compact group has an infinite Helson set, provided the group contains an infinite abelian subgroup (see [7, p. 166]). This follows from the extension theorem for the Fourier algebra of a closed subgroup of a compact group [2, Section 8.6.4].

THEOREM 5. *Let* P *be a Helson set in a locally compact metrizable group* G. *If* $\mu \in M_0(P)$, *then* $\mu = 0$.

Proof. Let ϕ be a bounded Borel function on P. Let T_ϕ be defined on $M_0(P)$ by the relation $T_\phi(\nu) = \int_P \phi\, d\nu$ ($\nu \in M_0(P)$). Now T_ϕ is a continuous linear functional on $M_0(P)$. Since $M_0(P)$ can be identified with a closed subspace of $C^*_\rho(G)$ *via* the map $\nu \mapsto \rho(\nu)$, we can extend T_ϕ to $C^*_\rho(G)$. Thus there exists an $f \in B_\rho(G) \subset C^B(G)$ (where $B_\rho(G)$ is the dual space of $C^*_\rho(G)$) such that $\int_P \phi\, d\nu = \int_P f\, d\nu$ ($\nu \in M_0(P)$) [5, p. 192]. But $M_0(P)$ is a band, and therefore $\int_P \phi g\, d\nu = \int_P fg\, d\nu$ ($g \in C^B(G)$, $\nu \in M_0(P)$). Thus $\phi\, d\nu = f\, d\nu$. Now we proceed as in the abelian and compact cases. ■

COROLLARY 6. *If* G *is a locally compact, metrizable (nondiscrete) group, then* $A(G) \neq C_0(G)$.

Proof. Let U be a relatively compact open subset of G. Then $L^1(U) \neq \{0\}$. But if $A(G) = C_0(G)$, then $\overline{U}$ is a Helson set. ■

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