ON THE POISSON-STIELTJES REPRESENTATION FOR FUNCTIONS WITH BOUNDED REAL PART

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1. INTRODUCTION

The main purpose of this paper is to establish results that connect a Poisson-Stieltjes integral with boundary properties of the function it represents. A well-known theorem of Herglotz [5, p. 196] states that a function f holomorphic in $|z| < 1$ with positive real part has a Poisson-Stieltjes representation

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \, d\mu(t) + i \, \Im f(0), \tag{1.1}$$

where μ is a nondecreasing function of bounded variation on $[-\pi, \pi]$. Briefly, we shall say that f is a *Herglotz* function with *mass distribution* μ if (1.1) holds. For such functions, Fatou's theorem [6, p. 46] shows that $\Re[f]$ has angular limit at e^{it_0} equal to $\mu'(t_0)$ wherever the derivative exists (including $\mu'(t_0) = +\infty$). We shall seek other relationships that connect f and its mass distribution μ.

THEOREM 1. *If* μ *is the mass distribution for a Herglotz function* f, *and* $\sup \Re[f] < \infty$, *then* μ *is nondecreasing and absolutely continuous and has bounded Dini derivates.*

In fact, every difference quotient of μ is bounded by the bounds on $\Re[f]$. Example 1 shows that f as distinguished from $\Re[f]$ may nevertheless be unbounded.

Further information about μ is obtained under the condition

$$\iint_G |f'(\sigma)|^2 \, d\sigma < \infty, \tag{1.2}$$

where G is a domain of the form $\{|z| < 1\} \cap \{|z - \zeta| < r\}$, $|\zeta| = 1$. The integral represents the area of f(G) on the Riemann surface associated with f. Condition (1.2) has been used extensively in the boundary theory of conformal mapping [1]. We shall say that f has the *finite-area property* at ζ ($|\zeta| = 1$) if (1.2) holds for some $r > 0$ (for brevity, we occasionally write $f \in \mathrm{FAP}(\zeta)$). The usefulness of this condition arises from the fact that if

$$G_n = \{|z| < 1\} \cap \{r_{n+1} < |z - \zeta| < r_n\}$$

and $r_n \to 0$ as $n \to \infty$, then

$$\iint_{G_n} |f'(\sigma)|^2 \, d\sigma = o(1) \quad (n \to \infty). \tag{1.3}$$

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Using (1.3), we obtain the following converse of Fatou's theorem.

THEOREM 2. *Let* f *be a Herglotz function with* sup $\Re[f] < \infty$ *and mass distribution* μ. *If* $\Re[f]$ *has asymptotic value* α *on some curve ending at* $\zeta = e^{it_0}$ *and* $f \in \mathrm{FAP}(\zeta)$, *then* $\mu'(t_0)$ *exists and equals* α.

Example 1 shows that Theorem 2 fails without the finite-area property.

The finite-area property imposes a certain symmetry upon μ. For instance, it implies that μ is *almost odd,* in the following sense.

THEOREM 3. *Let* f *be a Herglotz function whose mass distribution* μ *has the property*

$$\mu_\ell(0),\ \mu_r(0) > 0, \tag{1.4}$$

where $\mu_\ell(0)$ *and* $\mu_r(0)$ *denote the lower left and right Dini derivates of* $\mu(t)$ *at* $t = 0$. *If* sup $\Re[f] < \infty$ *and* $f \in \mathrm{FAP}(1)$, *then*

$$\lim_{t \to 0+} \frac{\mu(-t) - \mu(0)}{\mu(t) - \mu(0)} = -1\,. \tag{1.5}$$

It follows that $\mu^r(0) = \mu^\ell(0)$ and $\mu_r(0) = \mu_\ell(0)$. A variation of Example 1 shows the necessity of the finite-area property, while Example 2 shows that Theorem 3 fails without condition (1.4). The conclusion (1.5) of Theorem 3 suggests that μ should behave fairly well when the finite-area property is imposed. The function μ is said to be *smooth* [8] at $t = t_0$ if

$$\lim_{h \to 0} \frac{\mu(t_0 + h) + \mu(t_0 - h) - 2\mu(t_0)}{h} = 0\,.$$

The following is a simple consequence of Theorem 3.

THEOREM 4. *Let* f *be a Herglotz function with* sup $\Re[f] < \infty$ *and mass distribution* μ. *If* $f \in \mathrm{FAP}(1)$, *then* μ *is smooth at* $t = 0$.

Example 3 shows that smoothness does not imply the finite-area property.

The most obvious class of functions to which Theorem 4 applies is the class of schlicht, bounded functions in $|z| < 1$. For such functions, we summarize the above results as follows.

THEOREM 5. *Let* f *be a bounded schlicht function in* $|z| < 1$. *Then*

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t) + i\, \Im\, f(0), \tag{1.6}$$

where μ *is smooth and absolutely continuous on* $[-\pi, \pi]$ *and has bounded Dini derivates. Moreover, the Jordan decomposition of* μ *can be written*

$$\mu(t) = \alpha(t) - Mt\,, \tag{1.7}$$

where $M > 0$ *and* α *is nondecreasing.*

Example 4 shows that a representation of type (1.6) is not in general possible for unbounded schlicht functions.

2. HERGLOTZ FUNCTIONS

Let $f = u + iv$ be a Herglotz function. The mass distribution μ of f is determined [5, p. 196] by

$$\mu(t) = \lim_{r \to 1} \int_0^t u(r, \theta) d\theta + C, \tag{2.1}$$

where C is a constant. For definiteness, we shall always take $C = 0$.

Proof of Theorem 1. Equation (2.1) shows immediately that μ is nondecreasing and satisfies the periodicity relation $\mu(t + 2\pi) = \mu(t) + 2\pi u(0)$. Let $b = \inf \Re[f]$ and $B = \sup \Re[f]$. Then (2.1) and the mean-value theorem imply that

$$b \leq \frac{\mu(t) - \mu(t_0)}{t - t_0} \leq B. \tag{2.2}$$

Consequently, the Dini derivates of μ must be bounded by the bounds on $\Re[f]$. To show that μ is absolutely continuous, consider the Lebesgue decomposition

$$\mu = \alpha + \gamma + \sigma, \tag{2.3}$$

where α is absolutely continuous, γ is a continuous singular function, and σ is a saltus function. A lemma of Lohwater [3] shows that if σ is actually present in this decomposition, then $\Re[f]$ cannot be bounded. Thus we may take $\sigma(t) \equiv 0$ in (2.3). Since μ is nondecreasing, the same is true of α and γ. If γ is not constant, then (by a theorem of S. Saks [7, p. 128]) μ has infinite derivative on an uncountable set of points, and this contradicts the fact, implied by (2.2), that $\mu'(t)$ is finite wherever it exists. Thus we may set $\gamma(t) \equiv 0$ in (2.3), and the result follows.

Example 1. Let ω denote the harmonic measure in $|z| < 1$ of the upper semicircle. If ω^* denotes a conjugate function, then $h = \omega + i\omega^*$ is a schlicht mapping of $|z| < 1$ onto the vertical strip $\{0 < \Re[z] < 1\}$ [5, p. 33]. However, the mass distribution μ of h is given by

$$\mu(t) = \begin{cases} 0 & (-\pi \leq t \leq 0), \\ t & (0 \leq t \leq \pi). \end{cases}$$

Thus μ is nondecreasing and absolutely continuous, and it has bounded Dini derivates; therefore h is an unbounded Herglotz function. Moreover, h does not possess the finite-area property at $z = 1$, and $\mu'(0)$ does not exist. However, every value in [0, 1] is an asymptotic value of $\Re[h]$ at $z = 1$.

3. THE FINITE-AREA PROPERTY

The following lemma and its variations are basic to our investigation. The method of proof seems to have been employed first by W. Gross [2]. We use $m\{E\}$ to denote the Lebesgue measure of a set E.

LEMMA 1. *Let* f *be a Herglotz function with* $\sup \Re[f] < \infty$ *and mass distribution* μ. *Denote by* $E(\mu' > d)$ *the set of points in* $(-\pi, \pi)$ *where* $\mu'(t)$ *exists and exceeds the value* d. *If* $f \in$ FAP(1) *and there exists a sequence* $\{t_n\}$ *of real numbers decreasing to zero such that*

$$\liminf_{n\to\infty} \frac{m\{E(\mu' > d) \cap [0, t_n]\}}{t_n} = K > 0, \tag{3.1}$$

then there exists a sequence $\{\Gamma_n\}$ *of circular arcs with the following properties:*

(1) $\Gamma_n = \{|z| < 1\} \cap \{|z - 1| = r_n\}$,

(2) *if* $\tau_n = \frac{1}{2} Kt_n$, *then* $|1 - e^{i\tau_n}| \leq r_n \leq |1 - e^{it_n}|$,

(3) $f(\Gamma_n)$ *is rectifiable, and* length $f(\Gamma_n) \to 0$ *as* $n \to \infty$,

(4) *if* γ_n *is the endpoint of* Γ_n *in* $\Im z > 0$, *then* $\Re[f(\gamma_n)] > d$.

To prove Lemma 1, we first observe that

$$\liminf_{n\to\infty} \frac{m\{E(\mu' > d) \cap [\tau_n, t_n]\}}{t_n} \geq \frac{K}{2}. \tag{3.2}$$

Next, set

$$E_n = E(\mu' > d) \cap [\tau_n, t_n], \qquad R_n = \{\rho: \rho = |e^{it} - 1|, \ t \in E_n\},$$

$$G_n = \{z: |z| < 1, \ |z - 1| = \rho, \ \rho \in R_n\}.$$

From (3.2) we obtain the relations

$$\liminf_{n\to\infty} \frac{m\{R_n\}}{|1 - e^{it_n}|} = \liminf_{n\to\infty} \frac{m\{E_n\}}{t_n} \geq \frac{K}{2}. \tag{3.3}$$

Using Schwarz's inequality and (1.3), we see that

$$\begin{aligned} \left(\iint_{G_n} |f'(\sigma)|\, d\sigma\right)^2 &\leq \left(\iint_{G_n} |f'(\sigma)|^2 d\sigma\right)\left(\iint_{G_n} d\sigma\right) \\ &= o(1)\,|1 - e^{it_n}|^2 \qquad (n \to \infty). \end{aligned} \tag{3.4}$$

Now let

$$\Gamma(t) = \{|z| < 1\} \cap \{|z - 1| = |e^{it} - 1|\}, \qquad L(t) = \text{length } f(\Gamma(t)),$$

$$L_n = \inf\{L(t): t \in E_n\}.$$

Using polar coordinates centered at $z = 1$, we obtain the estimate

$$\iint_{G_n} |f'(\sigma)|\, d\sigma \geq L_n \int_{R_n} d\rho = L_n m\{R_n\}. \tag{3.5}$$

Combining (3.5), (3.4), and (3.3), we find that

$$L_n = o(1) \frac{|1 - e^{it_n}|}{m\{R_n\}} = o(1) \qquad (n \to \infty). \tag{3.6}$$

Thus we may select $p_n \in E_n$ and define $r_n = |1 - e^{ip_n}|$ with

$$\Gamma_n = \{|z| < 1\} \cap \{|z - 1| = r_n\},$$

so that

$$|1 - e^{i\tau_n}| \leq r_n \leq |1 - e^{it_n}|$$

and

$$\text{length } f(\Gamma_n) < L_n + 2^{-n}. \tag{3.7}$$

Since the endpoint $\gamma_n = e^{ip_n}$ of Γ_n in $\Im z > 0$ lies in the set $\{e^{it}: t \in E_n\}$, it follows from Fatou's theorem that $\Re[f]$ has asymptotic value $\mu'(p_n) > d$ on Γ_n at γ_n. Clearly, (3.6) and (3.7) imply that length $f(\Gamma_n) = o(1)$ as $n \to \infty$, and Lemma 1 is established.

Remark 1. We shall want to apply several variations of Lemma 1 that are evident from the above proof. For example, if the set $E(\mu' > d)$ is replaced by the set $E(\mu' < d)$, then the lemma holds, with result (4) altered to read $\Re[f(\gamma_n)] < d$. Another important variation occurs if we replace (3.1) by

$$\liminf_{n \to \infty} \frac{m\{E(\mu' > d) \cap [-t_n, 0]\}}{t_n} = K > 0.$$

In this case, result (4) is altered to the effect that the endpoint γ_n of Γ_n in $\Im z < 0$ has $\Re[f(\gamma_n)] > d$. A combination of these two variations clearly holds.

4. A CONVERSE OF FATOU'S THEOREM

We begin by establishing a lemma concerning absolutely continuous functions. The proof is followed by a remark indicating several variations.

LEMMA 2. *Let* α *be a nondecreasing absolutely continuous function on* $[-\pi, \pi]$ *with* $\alpha(0) = 0$. *Denote by* $E(a' > d)$ *the set of points* t *where* $\alpha'(t)$ *exists and has a value exceeding* d. *If* $\sup \alpha'(t) = B$ *and* $\frac{\alpha(\tau)}{\tau} > d$ *for some* τ *in* $(0, \pi)$, *then*

$$\frac{m\{E(\alpha' > d) \cap [0, \tau]\}}{\tau} \geq \frac{\frac{\alpha(\tau)}{\tau} - d}{B - d}. \tag{4.1}$$

To prove Lemma 2, note first that $B - d > 0$. Since the result holds trivially if $B = \infty$, we take $B < \infty$. Put

$$H = E(\alpha' > d) \cap [0, \tau].$$

Then

$$B\,m(H) + d(\tau - m(H)) \geq \int_0^\tau \alpha'(t)\,dt = \alpha(\tau);$$

from this the conclusion follows immediately.

Remark 2. We shall need the following variation of this lemma. *If* $\inf \alpha'(t) = b$ *and* $\frac{\alpha(\tau)}{\tau} < d$ *for some* τ *in* $(0, \pi)$, *then*

$$\frac{m\{E(a' < d) \cap [0, \tau]\}}{\tau} \geq \frac{d - \frac{\alpha(\tau)}{\tau}}{d - b}. \tag{4.2}$$

The proof is analogous to the one just given, and we omit it. There are obvious variations of Lemma 2 and Remark 2 that occur when the set $[0, \tau]$ is replaced by $[-\tau, 0]$. We omit the statement of these variations.

Proof of Theorem 2. We may obviously take $t_0 = 0$. We show that if $\mu'(0)$ does not exist when $f \in \mathrm{FAP}(1)$, then $\Re[f]$ cannot have an asymptotic value at $z = 1$. To begin, suppose that the right-hand Dini derivates $\mu_r(t)$ and $\mu^r(t)$ are distinct at $t = 0$, and define $\varepsilon = [\mu^r(0) - \mu_r(0)]/8$. Since $\mu^r(0) > \mu_r(0)$, there exist sequences $\{a_n\}$ and $\{b_n\}$, with $a_n, b_n > 0$ and $a_n, b_n \to 0$ as $n \to \infty$, such that

$$\frac{\mu(a_n)}{a_n} = \mu^r(0) - \varepsilon \quad \text{and} \quad \frac{\mu(b_n)}{b_n} = \mu_r(0) + \varepsilon$$

for all n. If we set $B = \sup \Re[f]$, then (4.1) and (4.2) of Lemma 2 and its variations imply that

$$\frac{m\{E(\mu' > \mu^r(0) - 2\varepsilon) \cap [0, a_n]\}}{a_n} \geq \frac{\varepsilon}{B + 2\varepsilon}$$

and

$$\frac{m\{E(\mu' < \mu_r(0) + 2\varepsilon) \cap [0, b_n]\}}{b_n} \geq \frac{\varepsilon}{B + 2\varepsilon}.$$

Lemma 1 and Remark 1 thus establish the existence of two sequences $\{\Gamma_n\}$ and $\{\Lambda_n\}$ of circular arcs with the following properties:

$$(1) \quad \begin{cases} \Gamma_n = \{|z| < 1\} \cap \{|z - 1| = r_n\}, \; r_n \to 0 \text{ as } n \to \infty, \\ \Lambda_n = \{|z| < 1\} \cap \{|z - 1| = \rho_n\}, \; \rho_n \to 0 \text{ as } n \to \infty, \end{cases}$$

(2) length $f(\Gamma_n)$ and length $f(\Lambda_n)$ both tend to 0 as $n \to \infty$,

(3) if γ_n and λ_n are the endpoints of Γ_n and Λ_n in $\Im z > 0$, then

$$\Re[f(\gamma_n)] > \mu^r(0) - 2\varepsilon \quad \text{and} \quad \Re[f(\lambda_n)] < \mu_r(0) + 2\varepsilon .$$

It follows that if $\Re[f]$ has asymptotic value α at $z = 1$, then simultaneously $\alpha \geq \mu^r(0) - 2\varepsilon$ and $\alpha \leq \mu_r(0) + 2\varepsilon$, which is impossible by the choice of ε.

If $\mu^\ell(0) > \mu_\ell(0)$ rather than $\mu^r(0) > \mu_r(0)$, a similar proof gives the result.

If μ has unequal right- and left-hand derivatives at $t = 0$, we adjust the proof as follows. Let $D_r(0)$ and $D_\ell(0)$ denote these one-sided derivatives, with $D_r(0) > D_\ell(0)$. We take $\varepsilon = [D_r(0) - D_\ell(0)]/8$ and determine sequences $\{a_n\}$ and $\{b_n\}$ with $a_n > 0$, $b_n < 0$, and $a_n, b_n \to 0$ as $n \to \infty$, such that

$$\frac{\mu(a_n)}{a_n} > D_r(0) - \varepsilon \quad \text{and} \quad \frac{\mu(b_n)}{b_n} < D_\ell(0) + \varepsilon .$$

The proof now proceeds much as that above. Conclusion (3) now must be altered to read

(3') if γ_n is the endpoint of Γ_n in $\Im z > 0$ and λ_n is the endpoint of Λ_n in $\Im z < 0$, then

$$\Re[f(\gamma_n)] > D_r(0) - 2\varepsilon \quad \text{and} \quad \Re[f(\lambda_n)] < D_\ell(0) + 2\varepsilon .$$

We see from Example 1 and standard properties of harmonic measure that this theorem fails without the finite-area property. A related result is given by L. H. Loomis [4].

Remark. The author wishes to thank the referee for shortening the proof of Lemma 2.

5. SYMMETRY OF THE MASS DISTRIBUTION

The finite-area property imposes a strong local symmetry on the mass distribution. Our first result in this direction is Theorem 3.

Proof of Theorem 3. Let $B = \sup \Re[f]$. From (2.2) and (1.4) we find that

$$\limsup_{t \to 0+} \frac{\mu(-t)}{\mu(t)} \leq -\frac{\mu_\ell(0)}{B} < 0 \quad \text{and} \quad \liminf_{t \to 0+} \frac{\mu(-t)}{\mu(t)} \geq -\frac{B}{\mu_r(0)} > -\infty .$$

Consequently, if (1.5) does not hold, there exists a sequence of points t_n decreasing to zero with the properties

$$\lim_{n \to \infty} \frac{\mu(-t_n)}{\mu(t_n)} = k \quad (-\infty < k < 0,\ k \neq -1), \qquad \lim_{n \to \infty} \frac{\mu(-t_n)}{-t_n} = L, \qquad \lim_{n \to \infty} \frac{\mu(t_n)}{t_n} = R .$$

Thus $L = |k| R \neq R$. Assuming, for example, that $L < R$ and $\varepsilon = \frac{R - L}{8}$, we may also require that each t_n satisfies the inequalities

$$(5.1) \qquad \frac{\mu(-t_n)}{-t_n} < L + \varepsilon, \qquad \frac{\mu(t_n)}{t_n} > R - \varepsilon .$$

Lemma 2 and the conditions (5.1) now imply that

$$\frac{m\{E(\mu' \gtrsim R - 2\varepsilon) \cap [0, t_n]\}}{t_n} > \frac{\varepsilon}{B + 2\varepsilon},$$

and Remark 2 implies that

$$\frac{m\{E(\mu' < L + 2\varepsilon) \cap [-t_n, 0]\}}{t_n} > \frac{\varepsilon}{B + 2\varepsilon} .$$

Applying Lemma 1, we now obtain two sequences $\{\Gamma_n\}$ and $\{\Lambda_n\}$ of arcs with the following properties

(1) $\Gamma_n = \{|z| < 1\} \cap \{|z - 1| = r_n\}, \quad \Lambda_n = \{|z| < 1\} \cap \{|z - 1| = \rho_n\}$,

(2) $|1 - e^{i\tau_n}| < r_n, \quad \rho_n < |1 - e^{it_n}|$,

(5.2) $$\lim_{n\to\infty} \frac{|1 - e^{it_n}|}{|1 - e^{i\tau_n}|} \leq 2\,\frac{B + 2\varepsilon}{\varepsilon},$$

(3) length $f(\Gamma_n)$ and length $f(\Lambda_n)$ tend to 0 as $n \to \infty$,

(4) if γ_n is the endpoint of Γ_n in $\Im z > 0$, and if λ_n is the endpoint of Λ_n in $\Im z < 0$, then f has asymptotic values at these points, and

(5.3) $$\Re[f(\gamma_n)] > R - 2\varepsilon, \qquad \Re[f(\lambda_n)] < L + 2\varepsilon.$$

Now let S be the triangular region with vertices at z = i, -i, and 1. Let S_n be the subregion of S bounded by subarcs of Γ_n and Λ_n. Because $f \in$ FAP(1), Schwarz's inequality implies that

(5.4) $$\left(\iint_{S_n} |f'(\sigma)|\,d\sigma\right)^2 \leq \left(\iint_{S_n} |f'(\sigma)|^2\,d\sigma\right)\left(\iint_{S_n} d\sigma\right)$$
$$= o(1)\,|1 - e^{it_n}|^2 \qquad (n \to \infty).$$

Now, for $\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$, define

$$\ell_n(\theta) = S_n \cap \{\arg(z-1) = e^{i\theta}\}, \quad L_n(\theta) = \text{length } f(\ell_n(\theta)), \quad \mathscr{L}_n = \inf_\theta L_n(\theta).$$

Using polar coordinates centered at z = 1, we obtain the inequality

(5.5) $$\iint_{S_n} |f'(\sigma)|\,d\sigma \geq \frac{\pi}{2}\,\mathscr{L}_n\,|1 - e^{i\tau_n}|.$$

From (5.5), (5.4), and (5.2) we conclude that

$$\mathscr{L}_n = o(1) \qquad (n \to \infty).$$

For each n there thus exists a radial segment R_n joining Γ_n and Λ_n, with length $f(R_n) \to 0$ as $n \to \infty$. Using R_n and the portions of Γ_n and Λ_n joining R_n to the points γ_n and λ_n, respectively, we obtain a curve C_n joining γ_n and λ_n, with length $f(C_n) \to 0$ as $n \to \infty$. But this is impossible, since (5.3) implies that

$$|f(\gamma_n) - f(\lambda_n)| > R - L - 4\varepsilon = 4\varepsilon.$$

Thus our assumption that (1.5) does not hold leads to a contradiction, and Theorem 3 is proved.

The function $H(z) = h(z) + 1$, where h is the function of Example 1, has mass distribution

$$\mu(t) = \begin{cases} t & (-\pi \leq t \leq 0), \\ 2t & (0 \leq t \leq \pi). \end{cases}$$

Thus μ satisfies (1.4), but $H \notin$ FAP(1), and condition (1.5) fails.

Example 2. We show now that Theorem 3 fails if we omit (1.4) from its hypotheses. Define a mass distribution μ on $[-\pi, \pi]$ as follows:

$$\mu(t) = \begin{cases} 0 & (-\pi \leq t \leq 0), \\ 1 - \cos t & (0 \leq t \leq \pi) . \end{cases}$$

This mass distribution generates a Herglotz function s that does not possess property (1.4), and the result (1.5) fails. We note in passing that s is bounded and schlicht. Boundedness follows from a direct computation of the conjugate $v(r, \theta)$ of $\Re[s]$:

$$v(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2r \sin t}{1 - 2r \cos t + r^2} \mu'(\theta - t)dt.$$

Univalence follows from the fact that $u = \Re[s]$ is continuous in $|z| \leq 1$. For if s were not schlicht, then the Riemann surface of s would contain two disjoint, noncompact arcs Γ_1 and Γ_2 projecting onto the *same* vertical line in the w-plane. Considering the boundary values of u, one sees that the images γ_1 and γ_2 in $|z| < 1$ of Γ_1 and Γ_2 have common endpoints on $|z| = 1$, hence bound a domain in which $\Re[s]$ is constant. But s is not constant, and therefore s must be schlicht.

Proof of Theorem 4. We have the representation

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t).$$

Hence, for $\varepsilon > 0$,

$$g(z) = f(z) + \varepsilon = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d(\mu(t) + \varepsilon t) \tag{5.6}$$

is a Herglotz function satisfying the conditions of Theorem 3. If $\alpha(t) = \mu(t) + \varepsilon t$, then for $t \neq 0$

$$\frac{\alpha(t) + \alpha(-t)}{t} = \frac{\alpha(t)}{t}\left(1 + \frac{\alpha(-t)}{\alpha(t)}\right).$$

Hence

$$\lim_{t \to 0} \frac{\alpha(t) + \alpha(-t)}{t} = 0,$$

so that α is smooth at $t = 0$. But then μ must also be smooth at $t = 0$, and the proof is complete.

Example 3. The function

$$f(z) = \exp\left(-\frac{1 + z}{1 - z}\right) + 1$$

is a bounded Herglotz function that does not possess the finite-area property at $z = 1$. The associated mass distribution is determined from (2.1) to be

$$\mu(t) = \int_0^t \cos\left(\cot\frac{t}{2}\right) dt + t .$$

This function has a finite derivative *everywhere,* hence is everywhere smooth.

6. APPLICATION TO SCHLICHT FUNCTIONS

Proof of Theorem 5. Most of the details here follow directly from the preceding facts and the observation that f possesses the finite-area property at *every* point of $|z| = 1$. The decomposition (1.7) follows from a consideration of $g(z) = f(z) + M$, where $M \geq \sup|f(z)|$. We omit the details.

Example 4. The function $w(z) = i\,\frac{1+z}{1-z}$ maps $|z| < 1$ onto the upper half plane. Since

$$\lim_{r \to 1} \int_0^{2\pi} |\Re[w]|\, d\theta = \infty ,$$

a Poisson-Stieltjes representation is not possible [5, p. 197].

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