REPRESENTATIONS OF INTEGRAL RELATION ALGEBRAS

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The main object of this note is to prove the following theorem.

The class of relation algebras that possess representations over a group is not finitely axiomatizable relative to the class of representable, integral relation algebras.

Previously it was not known whether the two classes are distinct. (The question had been stated as an open problem in [2].) To prove the theorem, we shall define and study an intermediate class of relation algebras. Roger Lyndon suggested an appropriate generic name for these algebras: *permutational.* An algebra will be called permutational if one of its representations admits a transitive group of automorphisms. Probably the most important unsolved problem related to our work is the question whether every representable, integral relation algebra is permutational. We shall strengthen some results of R. C. Lydon's paper [5] to obtain a negative solution of this problem under the assumption that there exists a finite projective plane whose order is not a power of a prime integer.

The final section contains the presentation of a nonrepresentable relation algebra having the smallest possible size.

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1. PRELIMINARIES

A *relation algebra* is a universal algebra of the type $\mathfrak{A} = \langle A, +, \cdot, -, ;, ^\smile, 1' \rangle$ that satisfies certain postulates due to Tarski (see for example [3, Definition 4.1]): $\langle A, +, \cdot, - \rangle$ is a Boolean algebra, the formulae

$$(x ; y) ; z = x ; (y ; z) \qquad \text{and} \qquad x ; 1' = 1' ; x = x$$

hold for all $x, y, z \in A$, and the formulae

$$(x ; y) \cdot z = 0, \qquad (x^\smile ; z) \cdot y = 0, \qquad (z ; y^\smile) \cdot x = 0$$

are equivalent for all $x, y, z \in A$. We use the symbols 0 and 1 to denote the Boolean null and unit element of $\mathfrak{A}$. A relation algebra is *representable* if it is isomorphic to an algebra $\mathfrak{D} = \langle D, \cup, \cap, \sim, |, ^\smile, I \rangle$, where, for some set X, $\langle D, \cup, \cap, \sim \rangle$ is a Boolean algebra of subsets of $X \times X$ (whose unit set is not necessarily equal to $X \times X$), where I is the identity relation on X, and where for any R and S belonging to D, $R | S$ is the relative product of R and S and $R^\smile$ is the converse of R. A representation of $\mathfrak{A}$ over the set X is a pair $\langle \mathfrak{D}, \phi \rangle$, where $\mathfrak{D}$

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is a concrete algebra of binary relations over X, as above, and ϕ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{D}$.

A relation algebra $\mathfrak{A}$ is *integral* if $\mathfrak{A}$ has more than one element and $x = 0$ or $y = 0$ for all $x, y \in A$ for which $x ; y = 0$. An integral relation algebra is simple (in the sense of universal algebra). Hence it is easy to see that if such an algebra is representable, then it has a representation $\langle \mathfrak{D}, \phi \rangle$ with unit set $1\phi = X \times X$. From now on, "representation" will mean representation in this restricted sense. A relation algebra is *representable over a group* if it is isomorphic to a subalgebra of an algebra

$$\mathfrak{A}(G) = \langle P(G), \cup, \cap, \sim, \cdot, {}^{-1}, \{e\} \rangle ,$$

called the *complex algebra* of G, and composed of all subsets of a group G whose neutral element is e. Here $R \cdot S$ is the complex product of R and S, and $R^{-1} = \{x^{-1}: x \in R\}$. In this paper, $\mathscr{I}$ will denote the class of all representable integral relation algebras, and $\mathscr{G}$ will be used for the class of all algebras representable over a group.

One can easily prove that $\mathscr{G} \subseteq \mathscr{I}$. To proceed further we introduce the following notions. The automorphism group $\operatorname{Aut}(\mathfrak{D}, \phi)$ of a representation over X is the group of all permutations σ of X such that each member of $\mathfrak{D}$ is left invariant by σ, in other words,

$$\langle x, y \rangle \in R \Leftrightarrow \langle x\sigma, y\sigma \rangle \in R$$

for all $x, y \in X$ and $R \in D$. For any transitive group K of permutations of a set X, the set of all subsets of $X \times X$ that are invariant under every member of K forms a relation algebra $\mathfrak{A}(K ; X)$. This algebra is integral, since the domain and the range of each nonempty invariant relation are equal to X.

We call a relation algebra $\mathfrak{A}$ *permutational*—and we write $\mathfrak{A} \in \mathscr{P}$ —if it is isomorphic to a subalgebra of some algebra $\mathfrak{A}(K ; X)$ (with K transitive); in other words, if $\mathfrak{A}$ possesses a representation whose automorphism group is transitive.

We shall prove that $\mathscr{G} \subseteq \mathscr{P} \subseteq \mathscr{I}$. The second inclusion is trivial, since every subalgebra of an algebra in $\mathscr{I}$ belongs to $\mathscr{I}$. To prove the first inclusion it is convenient to define an equivalent construction of permutational algebras. Let G be an arbitrary group, and let H be one of its subgroups. Put

$$A(G, H) = \{S \subseteq G: H \cdot S \cdot H = S\} .$$

Although it may not contain the member $\{e\} \in \mathfrak{A}(G)$, this set is closed under the remaining operations of the complex algebra of G. One sees that

$$\mathfrak{A}(G, H) = \langle A(G, H), \cup, \cap, \sim, \cdot, {}^{-1}, H \rangle$$

is an integral relation algebra. Let us call it the *algebra of double cosets* of H in G. Of course, if we take $H = \{e\}$, then $\mathfrak{A}(G, H) = \mathfrak{A}(G)$.

LEMMA 1.1. *Let* G *and* H *be groups such that* $H \subseteq G$. *Let* $X = \{H \cdot g: g \in G\}$, *and let* K *be the permutation group on* X *consisting of the right translations* $H \cdot g \to H \cdot g \cdot g_0$ $(g_0 \in G)$. *Then*

$$\mathfrak{A}(G, H) \cong \mathfrak{A}(K ; X) .$$

Conversely, let K *be a transitive permutation group on a set* X, *let* $x \in X$, *and let* $K_x = \{\sigma \in K: x\sigma = x\}$. *Then*

$$\mathfrak{A}(K ; X) \cong \mathfrak{A}(K, K_x).$$

Proof. The isomorphism to establish the first conclusion correlates each $S \in A(G, H)$ with the set $\{\langle H \cdot s \cdot g, H \cdot g \rangle: s \in S \text{ and } g \in G\}$. The isomorphism for the second statement maps each relation R, invariant under K, onto the set $\{\sigma: \langle x, x\sigma \rangle \in R\} \subseteq K$.

LEMMA 1.2. *Let* $\mathfrak{A}$ *be a relation algebra. A necessary and sufficient condition for* $\mathfrak{A} \in \mathscr{G}$ *is that* $\mathfrak{A}$ *possesses a representation* $\langle \mathfrak{D}, \phi \rangle$ *over a set* X *such that* $\mathrm{Aut}(\mathfrak{D}, \phi)$ *contains a subgroup* K *that is regular and transitive on* X.

Proof. We can restate the condition as $\mathfrak{A} \cong \mathfrak{D} \subseteq \mathfrak{A}(K ; X)$ for some set X and some group K regular and transitive on X. That K is regular means that $K_x = \{e\}$ for all $x \in X$. By Lemma 1.1 this implies that $\mathfrak{A}(K ; X) \cong \mathfrak{A}(K, \{e\}) = \mathfrak{A}(K)$. Thus the sufficiency is established. The necessity of the condition follows likewise from Lemma 1.1 and the fact that every group is isomorphic to a regular transitive permutation group.

LEMMA 1.3. *Let* G, H, *and* K *be groups with* $H \subseteq G$, $K \subseteq G$, *and* $H \cdot K = G$. *Then* $\mathfrak{A}(G, H)$ *is isomorphic to a subalgebra of* $\mathfrak{A}(K, H \cap K)$. *Furthermore, if* L *is a normal subgroup of* G *and* $L \subseteq H$, *then* $\mathfrak{A}(G, H) = \mathfrak{A}(G/L, H/L)$. *Thus, if* H *is a normal or a complemented subgroup of* G, *then* $\mathfrak{A}(G, H) \in \mathscr{G}$.

Proof. Let G, H, and K be given, with $H \cdot K = G$. Let

$$\phi = \langle S \cap K: S \in A(G, H) \rangle.$$

To verify that ϕ is an embedding of $\mathfrak{A}(G, H)$ into $\mathfrak{A}(K, H \cap K)$, let $R, S \in A(G, H)$. It should be clear from the definitions that $R\phi \in A(K, H \cap K)$, that $R^{-1}\phi = (R\phi)^{-1}$, and that $H\phi = H \cap K$. Of course, $(R \cdot S)\phi \supseteq R\phi \cdot S\phi$. But conversely, if $x = r \cdot s \in (R \cdot S)\phi$, that is, if $x \in K$, $r \in R$, $s \in S$, then we may write $s = h \cdot k$, where $h \in H$ and $k \in K$. Since $R \cdot H = R$ and $H \cdot S = S$, we see that $r \cdot h \in R$ and $k \in S$. Thus $x = (r \cdot h) \cdot k$, where $r \cdot h = x \cdot k^{-1} \in R \cap K$ and $k \in S \cap K$. In other words, $x \in R\phi \cdot S\phi$. Of course, ϕ is a homomorphism for the Boolean operations. To prove that it is one-to-one, we need only show that $R \neq 0$ implies $R\phi \neq 0$. This follows from the relations $H \cdot R = R$ and $H \cdot K = G$.

Now, if L is a normal subgroup of G and $L \subseteq H$, then we put

$$\psi = \langle \{Lr: r \in R\}: R \in A(G, H) \rangle,$$

and we verify in a similar manner that $\psi: \mathfrak{A}(G, H) \cong \mathfrak{A}(G/L, H/L)$.

That H is a complemented subgroup of G means that there exists a subgroup K such that $H \cdot K = G$ and $H \cap K = \{e\}$. Thus the final statement of the lemma follows from the first and the second.

2. AN EXAMPLE

We can now answer in the negative the question of Jónsson and Tarski [2] whether $\mathscr{G} = \mathscr{I}$. In fact, every algebra representable over a group obviously satisfies the universal implication $x ; x^{\smile} \leq 1' \rightarrow x^{\smile} ; x \leq 1'$. Let $\mathfrak{A} = \mathfrak{A}(G, H)$, a double-coset algebra, and suppose that G contains an element g satisfying the condition $g^{-1} \cdot H \cdot g > H$. We put $X = HgH \in \mathfrak{A}$, so that $X^{-1} \cdot X > H = X \cdot X^{-1}$; thus $\mathfrak{A} \notin \mathscr{G}$ if such an element g exists. As an example of such groups G and H and such an element $g \in G$, let G be the group of all permutations of the set of integers, let H be the subgroup fixing the nonnegative integers, and let g be the successor function $(ng = n + 1)$.

From our observations and Lemma 1.1 we conclude that $\mathscr{G} \subset \mathscr{P} \subseteq \mathscr{I}$.

3. THE MAIN RESULT

Although the definitions of the three classes introduced in Section 1 provide no means to actually determine whether an abstractly defined relation algebra belongs to a class, it is known that each of these classes is elementary in the wider sense: each is characterized by the satisfaction of a set of sentences formulated in the elementary (first-order) language of relation algebras. In fact, it was shown by A. Tarski for $\mathscr{G}$ and $\mathscr{I}$ (see [9]), and it follows likewise for $\mathscr{P}$, that each of the classes is characterized as the class of all integral relation algebras satisfying a certain set of axioms in the form of equations. (A specific set of equations characterizing the class of all representable algebras (not necessarily integral) was published by Lyndon in [4].)

In his paper [7], Monk showed that no class $\mathscr{K}$ such that $\mathscr{G} \subseteq \mathscr{K} \subseteq \mathscr{I}$ can be defined by a finite set of elementary sentences. In this section, we shall prove that there exists no finite set Δ of elementary sentences by which $\mathscr{G}$ could be characterized as the subclass of $\mathscr{I}$ consisting of the algebras that satisfy Δ. This is part of the content of Theorem 3.2—which follows directly from Theorem 3.1 by virtue of the fundamental theorem that every elementary sentence is preserved under the formation of ultraproducts (Theorem 5.1 of [6]).

THEOREM 3.1. *There exists an ultraproduct* $\mathfrak{A}$ *of algebras* $\mathfrak{A}_p \in \mathscr{P} \sim \mathscr{G}$ *such that* $\mathfrak{A} \in \mathscr{G}$.

THEOREM 3.2. *If* $\mathscr{K}$ *is an elementary class (in the wider sense) containing* $\mathscr{P}$, *then* $\mathscr{G}$ *is not finitely axiomatizable relative to* $\mathscr{K}$.

It would certainly be of interest to have a reasonably elegant system of first-order axioms characterizing $\mathscr{I}$, $\mathscr{P}$, or $\mathscr{G}$. In this direction we contribute the following observation:

PROPOSITION 3.3. $\mathscr{P}$ *is the class of all integral relation algebras satisfying every equation that is true in every complex algebra of a group and does not involve the symbol* 1'.

The proof of the proposition uses standard techniques of model theory, together with Tarski's method of converting every universal sentence into an equation that is equivalent to it in integral relation algebras [9]. Letting $\mathfrak{A}$ be an integral relation algebra satisfying the mentioned equations, we see that there exists a group G and an algebra $\mathfrak{B} \cong \mathfrak{A}$ such that $\langle B, +, \cdot, -, ;, ^{\smile} \rangle$ is a subalgebra of $\mathfrak{A}(G)$ when the nullary operation 1' is disregarded. If H = 1' in $\mathfrak{B}$, then we find that H is a subgroup of G and that $\mathfrak{B} \subseteq \mathfrak{A}(G, H)$. Hence, by Lemma 1.1, $\mathfrak{A} \cong \mathfrak{B} \in \mathscr{P}$. The converse assertion that every permutational algebra satisfies the mentioned equations is clear, in the light of Lemma 1.1.

Now, to begin the proof of Theorem 3.1, we first state an easy lemma. For each $t \in T$, where T is a nonempty set, let G_t be a group. Let D be an ultrafilter over

T. If X_t is a subset (or a subgroup) of G_t, for each $t \in T$, then the ultraproduct $\prod_{t \in T} X_t/D$ is not, strictly speaking, a subset of the group $\prod_{t \in T} G_t/D$. However, it can be identified with the subset (or subgroup, respectively) consisting of the f/D such that $\{t \in T: f_t \in X_t\} \in D$. We make this identification in the following assertion.

LEMMA 3.4. *Let* $G = \prod_{t \in T} G_t/D$ *and* $H = \prod_{t \in T} H_t/D$, *where* H_t *and* G_t ($H_t \subseteq G_t$) *are groups for each* $t \in T$. *Then there exists an algebra* $\mathfrak{B} \subseteq \mathfrak{A}(G, H)$ *such that* $\prod_{t \in T} \mathfrak{A}(G_t, H_t)/D \cong \mathfrak{B}$.

In fact, it is easily seen that the formula

$$(\langle X_t: t \in T \rangle /D)\phi = \prod_{t \in T} X_t/D$$

defines a map ϕ that is an isomorphism from $\prod \mathfrak{A}(G_t, H_t)/D$ into $\mathfrak{A}(G, H)$.

For the remainder of this section, D will denote a fixed, nonprincipal ultrafilter over the set P of all odd prime natural numbers. We define below two systems of groups H_p and G_p ($H_p \subseteq G_p$) for every $p \in P$. Let H and G be the respective ultraproducts of these systems. The following two lemmas will be proved.

LEMMA 3.5. *For each* p, $\mathfrak{A}(G_p, H_p) \notin \mathscr{G}$.

LEMMA 3.6. H *is a complemented subgroup of* G.

From Lemmas 3.6 and 1.3 it will follow that $\mathfrak{A}(G, H) \in \mathscr{G}$; hence, by Lemma 3.4, $\prod_{p \in P} \mathfrak{A}(G_p, H_p)/D \in \mathscr{G}$, although each algebra $\mathfrak{A}(G_p, H_p) \in \mathscr{P} \sim \mathscr{G}$ by Lemma 3.5. Thus Theorem 3.1 will be proved.

Let now p be any odd prime. The group G_p to be defined is a certain split extension of the dihedral group of order $2 \cdot p^2$ by the cyclic group Z_{p^2} of order p^2. (Thus it has order $2 \cdot p^4$.) We define it concretely as follows: The elements of G_p are the ordered triples (i, ε, j) of integers, with equality defined by $(i, \varepsilon, j) = (k, \gamma, \ell)$ if and only if $i \equiv k \pmod{p^2}$, $\varepsilon \equiv \gamma \pmod 2$, and $j \equiv \ell \pmod{p^2}$. Multiplication in the group is defined by the formula

$$(i, \varepsilon, j) \cdot (k, \gamma, \ell) = (i + k + 2(j\gamma - \varepsilon k) - 4\varepsilon j\gamma, \varepsilon + \gamma, j + \ell) \tag{3.7}$$

whenever $(i, \varepsilon, j), (k, \gamma, \ell) \in G_p$ and $0 \leq \varepsilon, \gamma < 2$. Thus (0, 0, 0) is the neutral element of the group, and

$$(i, \varepsilon, j)^{-1} = (4\varepsilon j - i, \varepsilon, -j). \tag{3.8}$$

Define H_p to be the cyclic subgroup of G_p whose elements are (0, 0, ip) ($0 \leq i < p$); and let L_p be the subgroup consisting of all elements of the form (0, 0, m), so that $H_p \subset L_p \subset G_p$.

Proof of Lemma 3.6. Our convention implies that $H \subseteq L \subseteq G$, where $L = \prod_{p \in P} L_p/D$. L is an abelian group and H is a divisible abelian group; this follows from Theorem 5.1 of [6] and the fact that for each integer $n > 0$ all but finitely many groups H_p satisfy the sentence $\forall x \exists y\ (y^n = x)$. A well-known theorem concerning abelian groups implies that H is complemented in L; therefore we can choose a group $H' \subseteq L$ so that $H \cdot H' = L$ and $H \cap H' = \{e\}$.

To complete the proof we simply remark that it is easy to show that the elements $\langle (i_p, \varepsilon_p, j_p)\colon p \in P\rangle/D$ of G such that $\langle (0, 0, j_p)\colon p \in P\rangle/D$ belongs to H' constitute a subgroup K satisfying the requirements $H \cdot K = G$ and $H \cap K = \{e\}$.

Proof of Lemma 3.5. Let p be an odd prime. If x belongs to some relation algebra, let x^p denote the p-fold relative power $x ; x ; \cdots ; x$. If x, y, and $x_1, \cdots, x_p$ are elements of the complex algebra of a group K, then the conditions

$$x^p = 1', \qquad y \neq 0, \qquad y ; x ; y^{\smile} = \sum_\alpha x_\alpha$$

jointly imply that $x_\alpha^p \geq 1'$ for some α. (In fact the first condition implies that $x = \{g\}$, where $g \in K$ and $g^p = e$; hence some x_α contains an element $k = h \cdot g \cdot h^{-1}$ $(h \in y)$ such that $k^p = e$.)

We proceed to show that the above implication fails in $\mathfrak{A}(G_p, H_p)$, from which we infer that this algebra cannot be embedded in a complex algebra.

In $A(G_p, H_p)$ consider the elements

$$R = H_p \cdot (0, 0, 1) \cdot H_p, \qquad S = H_p \cdot (0, 1, 0) \cdot H_p, \qquad \text{and}$$

$$R_\alpha = H_p \cdot (2(p - \alpha p - 1), 0, 1) \cdot H_p \text{ for } 1 \leq \alpha \leq p .$$

Using formulas (3.7) and (3.8), we find that

$$R^p = H_p \cdot (0, 0, p) \cdot H_p = H_p$$

(because (0, 0, 1) belongs to the centralizer of H_p and its pth power belongs to H_p), and we note that the last member is the relative identity element of $\mathfrak{A}(G_p, H_p)$. Also,

$$\begin{aligned} S \cdot R \cdot S^{-1} &= H_p \cdot (0, 1, 0) \cdot H_p \cdot (0, 0, 1) \cdot H_p \cdot (0, 1, 0) \cdot H_p \\ &= H_p \cdot (0, 1, 0) \cdot H_p \cdot (0, 0, 1) \cdot (0, 1, 0) \cdot H_p \\ &= \bigcup_{i=0}^{p-1} H_p \cdot (0, 1, 0) \cdot (0, 0, ip) \cdot (2, 1. 1) \cdot H_p \\ &= \bigcup_{i=0}^{p-1} H_p \cdot (-2ip - 2, 0, 1) \cdot (0, 0, ip) \cdot H_p = \sum_\alpha R_\alpha . \end{aligned}$$

Finally, to conclude the argument we show that $R_\alpha^p \not\geq H_p$ for each α. Note that (3.7) implies that $(2 \cdot (p - \alpha p - 1), 0, 1)$ belongs to the centralizer of H_p. Thus $R_\alpha^p = H_p \cdot (2p(p - \alpha p - 1), 0, p)$. Since $2p(p - \alpha p - 1) \not\equiv 0 \pmod{p^2}$, the desired result follows.

4. LYNDON ALGEBRAS

In this section we consider the question whether $\mathscr{P} = \mathscr{I}$. This appears to be a difficult problem; nevertheless we can give a sufficient condition for the answer to be negative.

Lyndon [5] defined an important class of integral relation algebras derived from finite projective planes. The Lyndon algebra $\mathfrak{A}_m$ $(m \geq 3)$ has 2^{m+2} elements and $m+2$ atoms, 1' and $a_0, \cdots, a_m$. It is completely specified by the relations

$$a_\alpha^\smile = a_\alpha, \qquad a_\alpha ; a_\alpha = a_\alpha + 1', \quad \text{and}$$

$$a_\alpha ; a_\beta = \sum \{a_\gamma : \gamma \neq \alpha, \beta\} \quad \text{if } \alpha \neq \beta .$$

In [5], Lyndon studied the representations of his algebras. He proved that $\mathfrak{A}_m \in \mathscr{G}$ if and only if m is a power of a prime integer, and that $\mathfrak{A}_m \in \mathscr{I}$ if and only if there exists a projective plane of order m (that is, a projective plane with $m+1$ points on each line). In fact, he showed that there is a canonical correspondence between the affine planes of order m and the representations of the algebra $\mathfrak{A}_m$.

Making use of Lyndon's work and an old theorem of Frobenius on permutation groups we prove the following theorem, which implies that $\mathfrak{A}_m \in \mathscr{I} \sim \mathscr{P}$ in case m is not a prime power and there exists a projective plane of order m.

THEOREM 4.1. *If* m *is not a prime power, then* $\mathfrak{A}_m$ *is not permutational.*

Proof. Let the pair $\langle \mathfrak{D}, \phi \rangle$ be a representation of $\mathfrak{A}_m$ over a set X such that $\mathrm{Aut}(\mathfrak{D}, \phi) = G$ acts transitively on X. We shall see that G contains a regular transitive subgroup. Then, by Lemma 1.2, $\mathfrak{A}_m \in \mathscr{G}$, and by Corollary 2.3 of [5], m is a power of a prime.

For $0 \leq \alpha \leq m$, let us put $E_\alpha = (1' + a_\alpha)\phi$, where the a_α are the atoms of $\mathfrak{A}_m$ other than 1'. From the defining relations of $\mathfrak{A}_m$ we deduce that the E_α are equivalence relations on X, and that $E_\alpha \mid E_\beta = X \times X$ and $E_\alpha \cap E_\beta = I$ (the identity relation on X) whenever $\alpha, \beta \leq m$ and $\alpha \neq \beta$. Furthermore,

$$\bigcup \{E_\alpha : \alpha \leq m\} = X \times X .$$

Now I claim that every transformation $\sigma \in G$ that fixes two distinct points x and y must fix all points. For let $\sigma \in G$ be such that the fixed set F of σ contains at least two points. If $x, y \in F$ and if $\langle x, u \rangle \in E_\alpha$ and $\langle y, u \rangle \in E_\beta$ $(\alpha \neq \beta)$ for some u, then $u \in F$. To see this, we observe that $\langle x, u\sigma \rangle \in E_\alpha$ and $\langle y, u\sigma \rangle \in E_\beta$, since the equivalence relations E_α and E_β are invariant under σ, and that therefore $\langle u, u\sigma \rangle \in E_\alpha \cap E_\beta$. Thus it follows that $u = u\sigma$.

Without losing generality, we may assume that $x, y \in F$, $x \neq y$, and $\langle x, y \rangle \in E_0$. Pick z so that $\langle x, z \rangle \in E_1$ and $\langle z, y \rangle \in E_2$ $(E_1 \mid E_2 = X \times X)$. By the argument above, $z \in F$. Now let $u \in X$. Choose $\alpha, \beta, \gamma \leq m$ so that $\langle x, u \rangle \in E_\alpha$, $\langle y, u \rangle \in E_\beta$, and $\langle z, u \rangle \in E_\gamma$. If any two of α, β, γ are distinct, then $u \in F$ by the above argument. But if $\alpha = \beta = \gamma$, then

$$\langle x, y \rangle \in E_0 \cap E_\alpha, \qquad \langle x, z \rangle \in E_1 \cap E_\alpha, \qquad \langle z, y \rangle \in E_2 \cap E_\alpha .$$

Then $\alpha = 0$, since $x \neq y$. This implies that $x = z = y$, since $E_1 \cap E_0 = E_2 \cap E_0 = I$. This contradiction shows that $u \in F$ and proves the claim.

I also claim that X is finite $(\overline{\overline{X}} = m^2)$. This follows from Theorem 1 of [5]; it can also be shown directly.

The desired result now follows from the two claims above and from the theorem of Frobenius (Theorem IV, p. 334 in [1]), which states that in a finite, transitive permutation group containing no nonidentity transformations that fix two or more elements, the identity element together with the transformations that move all points constitute a regular transitive subgroup.

5. EXAMPLES AND PROBLEMS

We now construct a "minimal" algebra that satisfies Tarski's axioms for relation algebras and fails to be representable. Our example has 16 elements and one can show that no such algebras with fewer than 16 elements exist. We do not know whether any other such algebras of this size exist. It is interesting to note that our example (and also the smallest previously known example, Lyndon's algebra $\mathfrak{A}_6$) is an integral algebra.

The Boolean part of the algebra $\mathfrak{C}$ is to be the subset algebra of a 4-element set; $C = P(4)$. Let the atoms of the algebra be $1'$, C_0, C_1, C_2. Then define

$$1'^{\smile} = 1', \qquad C_0^{\smile} = C_1, \qquad C_1^{\smile} = C_0, \qquad C_2^{\smile} = C_2,$$

and put

$$1' ; x = x ; 1' = x, \qquad C_0 ; C_0 = C_0, \qquad C_1 ; C_1 = C_1,$$

$$C_2 ; C_2 = {}^{\sim}C_2 \quad (= 1' \cup C_0 \cup C_1),$$

$$C_0 ; C_1 = C_1 ; C_0 = 1, \qquad C_\alpha ; C_2 = C_2 ; C_\alpha = C_2 \cup C_\alpha \quad (\alpha \in \{0, 1\}).$$

There exists exactly one way to extend these operations so that

$$\mathfrak{C} = \langle C, \cup, \cap, \sim, ;, \smile, 1' \rangle$$

is a relation algebra (; and $\smile$ must be additive operations). The verification that $\mathfrak{C}$ is in fact a relation algebra is routine.

Now, to obtain a contradiction, suppose that $\mathfrak{C}$ is representable. Since it is integral we then have a representation $\langle \mathfrak{D}, \phi \rangle$ where 1ϕ is of the form $X \times X$. If we let $R = C_0\,\phi$, the defining relations imply that R is a partial ordering of X under which every pair of elements has an upper and a lower bound, and two given elements are comparable if and only if they are both incomparable to some third element. It is a trivial matter to show that these assumptions are contradictory.

A final remark: the smallest example currently known by the author of a permutational algebra that is not representable over a group is a certain 7-atom subalgebra of the double-coset algebra $\mathfrak{A}(A_5, A_3)$, where A_5 and A_3 are the alternating groups. (The algebras $\mathfrak{A}(A_{n+2}, A_n)$ associated with alternating groups where $n > 3$ all belong to $\mathscr{G}$.)

Problem 1. Does $\mathscr{P} = \mathscr{G}$?

Problem 2. Consider the class of pairs $\langle G, H \rangle$ such that $\mathfrak{A}(G, H) \in \mathcal{G}$. These pairs can be construed as relational structures $\langle X, \cdot, {}^{-1}, P \rangle$ with two basic operations and a singulary relation P. Is this class elementary in the wider sense? Does it have any "constructive" characterization?

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