HEREDITARY LOCAL RINGS

Abraham Zaks

Professor P. M. Cohn recently provided an example of a right principal ideal domain with prime factorization that is not a left hereditary ring [1], and we were curious about the pathologies of a ring that allow this to happen.

In this note, we investigate some of the conditions under which a ring is a principal ideal domain. The treatment of the local case is easiest. Our results (see the lemma, and Corollaries 1 and 2) are closely related to those of G. A. Probert [5, Theorems 6.4 and 6.5]. We also obtain a result in the semilocal case (Proposition 1) that is similar to results obtained by G. O. Michler [3, Lemma 3.4 and Corollary 3.10]. (Our results are not obtainable from the results of Probert and Michler, nor do they imply them.) Our final remarks deal with the relations between principal ideal domains and rings all of whose proper residue rings are quasi-Frobenius.

By a *local ring* (R, M) we shall mean a ring R with a unique maximal ideal M that is both a left maximal ideal and a right maximal ideal.

By a *semilocal ring* we shall mean a ring R with finitely many maximal ideals $M_1, \cdots, M_t$, each being both a left maximal ideal and a right maximal ideal.

Recall that in an Ore domain, $Ra \cap Rb \neq 0$ whenever a and b are nonzero elements of R, and that a left principal ideal domain is both a left hereditary ring and a left Ore domain.

We shall also use freely the result that a projective module over a local ring is a free module [2].

1. THE SEMILOCAL CASE

PROPOSITION 1. *Let* R *be a semilocal ring. If* $M_i = Rm_i = m_i R$ *and* $\bigcap_{i=1}^{\infty} M_j^i = 0$ *for* $j = 1, \cdots, t$, *then* R *is a left and right principal ideal ring.*

Observe that the result is unambiguous, since the assumptions are symmetric.

If J is not nilpotent - in which case R is an Artinian ring - then R is a domain.

Proof. By straightforward reasoning, one finds that $M_i \cap M_j = M_i M_j$ for $i \neq j$ and $1 \leq i, j \leq t$. Also,

$$M_{j_1}^{i_1} \cap \cdots \cap M_{j_s}^{i_s} = M_{j_1}^{i_1} \cdot \cdots \cdot M_{j_s}^{i_s}$$

for every set $(i_1, \cdots, i_s)$ of nonzero integers whenever $p \neq q$ implies that $j_p \neq j_q$ $1 \leq j_1, \cdots, j_s \leq t$. Let

$$K = M_{j_1}^{i_1} \cap \cdots \cap M_{j_s}^{i_s}.$$

Received March 17, 1970.
This research was partially supported by National Science Foundation Grant GP-12546.

Michigan Math. J. 17 (1970).

We can easily show that R/K is an Artinian ring, by constructing a Jordan-Hölder series for R/K. It follows that R/K is isomorphic to $R/M_{j_1}^{i_1} \oplus \cdots \oplus R/M_{j_s}^{i_s}$, since R/K is an Artinian restricted quasi-Frobenius ring. Consequently, R/K is a principal ideal ring. In particular, the result follows in case J is a nilpotent ideal. Otherwise, let I be any left ideal. From the hypothesis $\bigcap_{i=1}^{\infty} M_j^i = 0$ it easily follows that $I \supset M_1^{i_1} \cdots M_t^{i_t}$ for suitable nonnegative integers $i_1, \cdots, i_t$ ($M_j^0 = R$). Therefore,

$$JI \supset M_1^{i_1+1} \cdots M_t^{i_t+1} .$$

Considering the ring $R/M_1^{i_1+1} \cdots M_t^{i_t+1}$, we see that there exists an element x in R such that

$$I = Rx + M_1^{i_1+1} \cdots M_t^{i_t+1} \subset Rx + JI .$$

Hence, if I is a finitely generated ideal, then $I = Rx$, whence $I = M_1^{j_1} \cdots M_t^{j_t}$ for suitable integers $j_1, \cdots, j_t$. The result for an arbitrary ideal follows now if we notice that Rx is a two-sided ideal and R/Rx is an Artinian ring.

A particular consequence is that every ideal of R is a two-sided ideal.

Finally, because J is not nilpotent, R must be a domain. In fact, it is an Ore domain.

It seems reasonable to ask whether Proposition 1 holds under the weaker assumption $\bigcap_{i=1}^{\infty} J^i = 0$, instead of $\bigcap_{i=1}^{\infty} M_j^i = 0$. This is probably so if the nonzero prime ideals of R are maximal ideals.

2. THE LOCAL CASE

LEMMA. *Let* (R, M) *be a local ring. Let* M *be a principal left ideal, and let* $\bigcap_{i=1}^{\infty} M^n = (0)$. *Then* R *is a left principal ideal domain, or else it is a left Artinian, left principal ideal ring.*

Proof. Let $M = Rm$; then $M^n = Rm^n$ for every integer n ($n \geq 1$). Let I ($I \neq (0)$) be any left ideal, and let $x \neq 0$ be an element in I. Since $\bigcap_{n=1}^{\infty} M^n = (0)$, it follows that $x \in M^u$ and $x \notin M^{u+1}$ for some integer u. Hence $x = rm^u$, $r \in R$, and $r \notin M$. Therefore r is invertible in R, and consequently $Rx = M^u$. This readily implies that $I = M^v = Rm^v$ for a suitable integer v. In particular, it follows that R is either left Artinian or that it is a domain. This completes the proof.

It is easy to verify that in a left principal ideal domain

$$\bigcap_{n=1}^{\infty} M^n = \left(\bigcap_{n=1}^{\infty} M^n\right) M ;$$

therefore the relation $\bigcap_{n=1}^{\infty} M^n = (0)$ is a consequence of either of the following two assumptions on the local ring R:

(a) R is right Noetherian,

(b) R is right hereditary.

However, under (a) it follows that R is right hereditary.

If R is not left Artinian (for example, if M is not nilpotent), then R is a Noetherian prime ring, and thus it has a simple ring as its right ring of quotients. If M were not principal as a right ideal, then R would contain direct sums with an arbitrarily large number of summands. However, if M is generated by α elements, then M^k is generated by $k\alpha$ elements. Therefore M is a principal right ideal. We have now established the following result.

COROLLARY 1. *Let* (R, M) *be a local ring, let* M *be a principal left ideal, and let* R *be a right Noetherian ring. If* M *is not a nilpotent ideal, then* R *is a left and right principal ideal domain.*

COROLLARY 2. *A right hereditary, right Noetherian local ring* (R, M) *is a right principal ideal domain.*

Proof. If M were nilpotent, it would follow that $M = 0$, since in this case R is a semiprimary ring. However, a semiprimary local ring is hereditary if and only if $M = 0$. Again, if M is free and R is a prime Goldie ring, then M is necessarily a principal right ideal. Moreover, the same type of argument applies to all right ideals.

Since a local, right hereditary, right Ore domain is right Noetherian, Corollary 2 may also be put as follows.

COROLLARY 3. *A local, right hereditary, right Ore domain is a right principal ideal domain.*

Notice that the condition that M is a left principal ideal is satisfied if either

(1) M/M^2 is a one-dimensional R/M-module and M is a finitely generated left R-module, or

(2) R is a left semihereditary Ore domain and M is a finitely generated module, or

(3) R is an Ore domain and M is a free module.

If (R, M) is a local left hereditary ring, then its center Z is a discrete valuation ring. In case Z is not a field, R is a left principal ideal domain (this was proved by P. M. Cohn in [1]). It is also a consequence of the following proposition.

PROPOSITION 2: *Let* (R, M) *be a local ring with center* Z. *Let* $0 \neq z \in M \cap Z$, *and let* M *be a left projective* R*-module. Assume that* $\bigcap_{i \in I} A_i$ *is either a projective module or the zero module, whenever* A_i *is a projective ideal for each* i *in* I. *Then* R *is a left principal ideal domain.*

(That the proposition implies Cohn's result is obvious.)

Proof. Since R is a local ring, we may replace "projective" by "free". We claim that if A and N are left free ideals and N is a right ideal, then NA is a left free ideal. Let $(a_\alpha)_{\alpha \in J}$ be a basis for A, and let $(n_\beta)_{\beta \in L}$ be a basis for the left free ideal N. Then the elements $n_\beta a_\alpha$ with $\beta \in L$ and $\alpha \in J$ form a basis for NA. To see this, observe that an element in NA is a sum of terms of the form na ($n \in N$, $a \in A$). Therefore

$$na = n \sum_{\alpha \in J} r_\alpha a_\alpha = \sum_{\alpha \in J} (nr_\alpha)a = \sum_{\alpha \in J}\Big(\sum_{\beta \in L} s_{\alpha\beta} n_\beta\Big) a_\alpha = \sum_{\substack{\alpha \in J \\ \beta \in L}} s_{\alpha\beta}(n_\beta a_\alpha)\,.$$

Obviously, the set of pairs of indices (α, β) for which $s_{\alpha\beta} \neq 0$ is finite. Assume

$$\sum_{\substack{\alpha \in J \\ \beta \in L}} u_{\alpha\beta} n_\beta a_\alpha = 0\,,$$

with only finitely many pairs (α, β) for which $u_{\alpha\beta} \neq 0$. Then

$$\sum_{\substack{\alpha \in J \\ \beta \in L}} u_{\alpha\beta} n_\beta a_\alpha = \sum_{\alpha \in J}\Big(\sum_{\beta \in L} u_{\alpha\beta} n_\beta\Big) a_\alpha = 0\,,$$

whence $\sum_{\beta \in L} u_{\alpha\beta} n_\beta = 0$, which in turn implies that $u_{\alpha\beta} = 0$ for all indices $\alpha \in J$ and $\beta \in L$. Therefore the set of elements $n_\beta a_\alpha$ with $\beta \in L$ and $\alpha \in J$ forms a basis for the free ideal NA.

In particular, since M is a free ideal, it follows that M^k is a free ideal, whence either $J = \bigcap_{i=1}^{\infty} M^i$ is a free ideal, or else $J = (0)$. We claim that $J = (0)$. Assuming that this is not the case, we shall obtain a contradiction.

Let z be a nonzero element in $Z \cap M$. Assume $z \in J$. If $z \in M^k J$ and $z \notin M^{k+1}J$, then M^kJ being a free ideal implies that we can choose z to be an element in a basis for M^kJ. In particular, z is a nonzero divisor. This in turn leads to the conclusion that M^kJ is a principal ideal, namely $M^kJ = Rz$. Since M is a free ideal, $J = JM$; thus $J^2 = JM^kJ = M^kJ^2$. This equality implies $J^2 = 0$, which contradicts the hypothesis $0 \neq z \in J$. As one verifies, $\bigcap_{k=1}^{\infty} M^k J = J^2$, and it follows that $z \in J^2$. A similar argument proves that $z \in J^m$ for every integer m. In particular, $z \in J_2 = \bigcap_{m=1}^{\infty} J^m$. Furthermore, J^m being free implies that J_2 is a free ideal. Also, $J_2 = J_2 M$. We set $J_1 = J$.

Similarly we find that $z \in J^k J_2$, whence $z \in \bigcap_{k=1}^{\infty} J^k J_2$. We set $J_3 = \bigcap_{k=1}^{\infty} J^k J_2$. Again, $J_3 M = J_3$, and J_3 is a free ideal.

We define J_α inductively for each ordinal α:

(1) if α is a limit ordinal, then $J_\alpha = \bigcap_{\beta < \alpha} J_\beta$, and

(2) if $\alpha = \beta + 1$, then $J_\alpha = \bigcap_{k=1}^{\infty} J^k J_\beta$.

As we argued earlier, one verifies that for every ordinal α

(a) $J_\alpha M = J_\alpha$,

(b) J_α is a free left ideal,

(c) $z \in J_\alpha$.

The ideals thus defined form a strictly decreasing sequence; indeed, for each ordinal $\delta < \varepsilon$, $J_\delta \supsetneqq J J_\delta \supseteqq J_\varepsilon$, since $J_\delta = J J_\delta$ implies $J_\delta = M J_\delta$; thus $J_\delta = (0)$, which contradicts (c).

But this is an impossible setting on the set R.

This contradiction is a result of assuming that $z \in J$. Therefore $z \notin J$. In particular, there exists an integer k such that $z \in M^k$ and $z \notin M^{k+1}$. As a consequence, $Rz = M^k$. From the relation $J = JM$ it easily follows that $J = JM^k$. However, $M^k = Rz$ implies $J = JM^k = M^k J \subset MJ \subset J$. Thus $J = MJ$ implies $J = (0)$. This completes the proof of the claim.

We have now proved that

(i) $M = Rm$, since M is free and $M^k = Rz$ is a principal free ideal,

(ii) $J = \bigcap_{i=1}^{\infty} M^i = (0)$.

The proposition now follows from the Lemma.

The Artinian case is ruled out in our setting, since it is impossible for M to be free in that case. Thus R is always a domain.

We observe that all the local rings we investigated turned out to be left (or right) Ore domains. The case of the non-Ore domain is attractive; however, it seems that its investigation requires some preparatory work. Also interesting seems the possibility of replacing the hypothesis of the Ore condition with the assumption that R is a subring of a division ring.

A different approach to the topic under consideration results from the point of view of quasi-Frobenius rings.

If R is a left-right Noetherian local ring (R, M) and R/M^2 is a quasi-Frobenius ring, then R is a right principal ideal ring, and either R is right Artinian or else R is a domain. Also, if R/I is a quasi-Frobenius ring whenever $I \neq 0$ is a two-sided ideal, then R is a right and left principal ideal domain whenever some nonzero element in the center of R belongs to M.

There naturally arises the following question:

If R is a left and right principal ideal domain, is R/I a quasi-Frobenius ring, for every nonzero, two-sided ideal I? That the answer is in the affirmative may be verified as follows. The inclusion relation between left (right) ideals induces a partial ordering of the left (right) ideals. Let $I = Rx$ be any nonzero left ideal. There exists a one-to-one order-reversing correspondence between left ideals Ry that contain Rx and right ideals aR, where $ay = x$. Consequently, R/Rx is an Artinian left R-module. Since a similar result holds for R/xR, we see that R/I is an Artinian ring whenever I is a nonzero ideal. Because being a left and right principal ideal ring is a property inherited by all the residue rings, it follows that every proper residue ring R/I of R is an Artinian principal ideal ring (left and right). Consequently, R/I is a quasi-Frobenius ring for every nonzero two-sided ideal I.

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Technion, I.I.T.
Haifa, Israel
and
Northwestern University
Evanston, Illinois 60201