THE TWO-GENERATOR PROBLEM FOR IDEALS

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Among the first things every student of Modern Algebra learns are the ways in which Dedekind rings are generalizations of principal ideal domains. Almost incidental is the fact that every ideal of a Dedekind ring can be generated by two elements. There are examples to show that this property in no way characterizes Dedekind rings. For instance, there is the ring of power series in one variable (over an arbitrary field) that have no linear term. There are no obvious ring-theoretic reasons to suggest that there should be some intrinsic way of distinguishing those rings in which every ideal can be generated by two elements from those requiring three, or four, or any number of generators. That the number two can be shown to be unique is somewhat remarkable.

For many years, the matter rested with the theorem of I. S. Cohen [4] that if there exists a fixed finite bound on the number of elements required to generate an ideal, then the ring is Noetherian and of Krull dimension one. The next step occurred when the author [6] proved that if a Noetherian local domain (subject to the somewhat crippling hypothesis that its integral closure is a finitely generated valuation ring) has the property that every ideal can be generated by two elements, then every ring extension between the domain and its integral closure is a Gorenstein ring. Shortly thereafter, H. Bass [2] proved that this property of Noetherian domains uniquely characterizes the two-generator property. However, Bass also needed a restrictive hypothesis; namely, that the integral closure is a finitely generated module. In addition, Bass proved that each of the conditions above is equivalent to the assertion that every indecomposable, finitely generated, torsion-free module is isomorphic to an ideal.

The theorem of this paper removes the restrictive hypotheses concerning the integral closure; and by replacing the concept of a Gorenstein ring of dimension one by the more general concept of a reflexive ring, our theorem generalizes the characterizations to arbitrary integral domains. Corollary 1 represents an even further sharpening of the theorem for the case where it is assumed that the domain is Noetherian. It must be said, however, that Bass considered Noetherian rings that have no nilpotent elements, but are not necessarily integral domains; hence his theorem is both more and less general than the one we present here.

We shall now proceed to give several definitions and explain some of our notation. Throughout this paper, R is an integral domain (not a field) with quotient field Q, and we set $K = Q/R$. If A is an R-module, we let $A^* = \mathrm{Hom}_R(A, R)$, the dual of A with respect to R. There exists a canonical map $A \to A^{**}$; if this map is a monomorphism, A is said to be *torsionless;* if the map is an isomorphism, A is said to be *reflexive.*

The integral domain R is called a *reflexive ring* if every torsionless R-module of finite rank is reflexive. We have shown in [9] that R is a reflexive ring if and only if K is the injective envelope of the direct sum of one copy of each of the simple R-modules. If R is a Noetherian domain, then R is a reflexive ring if and only if R

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is a Gorenstein ring of Krull dimension 1 (a Noetherian ring is a Gorenstein ring if the ring has finite injective dimension over itself).

If I is an R-submodule of Q, we let $I^{-1} = \{x \in Q \mid xI \subset R\}$. Then I^{-1} is naturally isomorphic to I^*; and the inclusion map $I \to I^{-1-1}$ is the canonical map $I \to I^{**}$. It has been shown (see [2] and [9]) that a Noetherian ring is a reflexive ring if and only if $I = I^{-1-1}$, for every ideal of R. A necessary condition for R to be a reflexive ring is that M^{-1} be generated by two elements, for every maximal ideal M of R. If R is Noetherian and of Krull dimension one, this condition is also sufficient. Moreover, if R is a Noetherian integral domain of Krull dimension 1 such that every maximal ideal can be generated by two elements, then R is a reflexive ring. For these results, see [9].

A *local ring* is a ring with only one maximal ideal, and a *semilocal ring* is a ring with only a finite number of maximal ideals (no Noetherian conditions are assumed). An h-*local ring* is a ring in which every nonzero element is contained in at most a finite number of maximal ideals, and every nonzero prime ideal is contained in only one maximal ideal. An h-local domain is a domain for which localizations behave properly; in other words, if T is a torsion R-module, then

$$T \cong \sum_M \oplus T_M,$$

where M ranges over all maximal ideals of R. In [9], we showed that a reflexive ring is an h-local ring.

An integral domain R is said to have *property* FD if every finitely generated torsion-free R-module is a direct sum of modules of rank 1. Equivalently, an integral domain has property FD if and only if every indecomposable, finitely generated, torsion-free R-module is isomorphic to an ideal of R. We say that R has *property* FD *locally* if R_M has property FD for every maximal ideal of R.

We define $\mu_*(R)$ to be the supremum of the minimum number of generators required to generate an ideal of R.

LEMMA. *Let* R *be an* h-*local ring. Then the following statements are true:*

(1) R *is a reflexive ring if and only if* R_M *is a reflexive ring, for every maximal ideal* M *of* R.

(2) R *is a Noetherian ring if and only if* R_M *is a Noetherian ring, for every maximal ideal* M *of* R.

(3) $\mu_*(R) = \max(2, k)$, *where* $k = \sup_M \mu_*(R_M)$ *and* M *ranges over all maximal ideals of* R (k *can be infinite*).

Proof. (1) If R is a reflexive ring, then R_M is a reflexive ring, by [9, Cor. 2.8]. Conversely, assume that R_M is a reflexive ring, for every maximal ideal M of R. Let $K_M = Q/R_M$; then, by [9, Cor. 2.6], K_M is the injective envelope of R_M/MR_M over R_M. Since $R_M/MR_M \cong R/M$, one can easily see, using [3, Chapter VI, Ex. 10], that K_M is the injective envelope of R/M over R. Since R is h-local, we have the relation

$$K = \sum_M \oplus K_M,$$

by [8, Theorem 3.1]. By [8, Theorem 3.3], K is an injective R-module. It is then clearly the injective envelope of $\sum_M \oplus R/M$ over R. By [9, Cor. 2.6], we have that R is a reflexive ring.

(2) and (3). If R is Noetherian, then it is elementary that R_M is Noetherian, for every maximal ideal M of R. Conversely, assume that R_M is Noetherian for every M. An examination of the proof of [1, Prop. 1.4] shows that all that is required for the proof to be valid is that a nonzero element of R be contained in only a finite number of maximal ideals of R. This establishes that R is Noetherian and proves (3) as well.

THEOREM. *Let* R *be an arbitrary integral domain with quotient field* Q. *Then the following three statements are equivalent:*

(1) *Every ideal of* R *can be generated by two elements.*

(2) *If* S *is a ring extension of* R *in* Q *that is finitely generated as an* R*-module, then* S *is a reflexive ring and* $\bigcap I^n = 0$, *for every ideal* I *of* S.

(3) R *is a Noetherian ring that has property* FD *locally. In this case,* R *is a Noetherian domain of Krull dimension* 1.

Proof. (1) $\Rightarrow$ (2). Assume that every ideal of R can be generated by two elements. I. S. Cohen has shown that R is then a Noetherian domain of Krull dimension 1. It follows from [9, Theorem 3.9] that R is a reflexive ring. For Noetherian domains, it is well known that

$$\bigcap_n I^n = 0,$$

for every ideal I of R.

If S is some finitely generated ring extension of R in Q, then every ideal of S is R-isomorphic to an ideal of R and hence can be generated by two elements over R, and a fortiori over S. Thus S inherits all properties of R listed above.

(2) $\Rightarrow$ (1). Let M be a maximal ideal of R. We shall show that R_M inherits the properties of R. Let A be a ring extension of R_M in Q that is finitely generated as an R_M-module. Then there exist elements $x_1, \cdots, x_n$ in A such that

$$A = R_M[x_1, \cdots, x_n].$$

Each x_i is integral over R_M, and hence there exist elements $s_1, \cdots, s_n$ in R - M such that $a_i = s_i x_i$ is integral over R ($i = 1, \cdots, n$). Let $S = R[a_1, \cdots, a_n]$; then $S_M = A$. It follows from [11, Chapter V, Section 1] that S is finitely generated as an R-module.

Since S is finitely generated over R, one can show that there exists a one-to-one correspondence between the maximal ideals of A and the maximal ideals of S that do not meet R - M. Hence, if L is a maximal ideal of A, there is a maximal ideal N of S such that $N_M = L$ and $L \cap S = N$. For each positive integer k, we have that

$$L^k = (N_M)^k = (N^k)_M.$$

Since N^k is a primary ideal for the maximal ideal N, we have the equation $\bigcap_k (N^k)_M = \left(\bigcap_k N^k\right)_M$. Now $\bigcap_k N^k = 0$, by assumption on S, and therefore

$$\bigcap_k L^k = \bigcap_k (N^k)_M = \left(\bigcap_k N^k\right)_M = 0.$$

It follows that $\bigcap_k I^k = 0$, for every ideal I of A.

Since S is a reflexive ring by assumption, it follows from [9, Theorem 2.7] that S is an h-local ring. By the one-to-one correspondence between the maximal (respectively, prime) ideals of A and the maximal (respectively, prime) ideals of S that do not meet $R - M$, we have that A is an h-local ring. One can easily see that $A_L = (S_M)_L = S_N$. Since S is a reflexive ring, S_N is a reflexive ring, by the lemma. Hence A is an h-local ring with the property that A_L is a reflexive ring, for every maximal ideal L of A. By the lemma, A is a reflexive ring.

Now that we see that R_M possesses all assumed properties of R, the lemma tells us that if (2) implies (1) for local domains, then (2) implies (1), for every integral domain. Hence we may assume from now on that R is a local ring with maximal ideal M.

If M is a projective ideal of R, then it is a principal ideal of R. Because of the condition $\bigcap_k M^k = 0$, this would imply that R is a discrete valuation ring. This would establish the implication (2) $\Rightarrow$ (1), and hence we may assume that M is not a projective ideal of R. But then $M^{-1}M = M$, which shows that $M^{-1} = R_1$ is a ring extension of R in Q. By [9, Prop. 2.5], R_1 is generated by two elements over R and $R_1/R \cong R/M$. Hence R_1 is a reflexive ring by assumption.

If I is not a principal ideal of R, then I is actually an ideal of R_1. For we have the inclusion $II^{-1} \subset M$, and therefore

$$(R_1 I)I^{-1} \subset R_1 M = M.$$

Hence $R_1 I \subset (I^{-1})^{-1} = I$.

We proceed to establish several properties of R and R_1.

(a) M *is a principal ideal of* R_1.

If I is an ideal of R_1, we define $I^{\#} = \{x \in Q \mid xI \subset R_1\}$; thus $I^{\#}$ is the dual of I with respect to R_1. Suppose M is not a principal ideal of R_1. Since R_1 is a finitely generated module over the local ring R, R_1 is a semilocal ring, and finitely generated projective R_1-modules are free. Thus M is not a projective ideal of R_1, and we have that $MM^{\#} \neq R_1$. Thus there exists a maximal ideal P of R_1 such that $MM^{\#} \subset P$. Hence

$$(P^{\#}M)M^{\#} \subset P^{\#}P \subset R_1,$$

and therefore $P^{\#}M \subset M^{\#\#}$. Since R_1 is a reflexive ring, we have that $M^{\#\#} = M$, and hence $P^{\#}M \subset M$. Thus $P^{\#} \subset M^{-1} = R_1$, and it follows that $P^{\#} = R_1$. But R_1 is a reflexive ring, and therefore $P^{\#\#} = P$. This contradiction shows that M is a principal ideal of R_1.

We have just shown that there exists an element $a \in M$ such that $M = R_1 a$. Thus we have the relations

$$M^2 = Ma \subsetneq Ra \subsetneq M.$$

It also follows that if $x \in M$, then $x/a \in R_1$.

(b) *If* R_1 *is not a local ring with maximal ideal* $N \supsetneqq M$, *then* R_1 *is a principal ideal ring.*

Since $R_1/R \cong R/M$, every maximal ideal of R_1 contains M, and there are at most two such ideals. If R_1 is local with maximal ideal M, then, since M is a principal ideal of R_1 and $\bigcap_k M^k = 0$, it follows that R_1 is a discrete valuation ring. Thus we may suppose that R_1 has two maximal ideals N_1 and N_2. Then $M = N_1 \cap N_2 = N_1 N_2$. It follows that N_1 and N_2 are projective ideals and thus principal ideals of R_1. Since

$$\bigcap_k N_1^k = 0 \quad \text{and} \quad \bigcap_k N_2^k = 0,$$

we see that N_1 and N_2 are the only nonzero prime ideals of R_1. By Cohen's Theorem [10, Chapter I, Theorem 3.4], R_1 is a Noetherian ring, and this forces R_1 to be a principal ideal ring, establishing our assertion.

If R_1 is a principal ideal ring, then every nonprincipal ideal of R, being an ideal of R_1, is isomorphic to R_1. Because R_1 is generated by two elements over R, every ideal of R can thus be generated by two elements. Therefore we may assume that R_1 is not a principal ideal ring. We then have, by (b), that R_1 is a local ring with maximal ideal $N \supsetneqq M$. If N is a principal ideal of R_1, then R_1 is a principal ideal ring. Thus we may assume that N is not a principal ideal of R_1. Let $R_2 = N^{\#}$, the dual of N with respect to R_1; then R_2 is a ring that is generated by two elements over R_1.

(c) $N = R_2 a$, *where* $M = R_1 a$.

We have the inclusions $M \subsetneqq N \subsetneqq R_1$; hence, by taking duals with respect to R, we have that $M \subsetneqq N^{-1} \subsetneqq R_1$. It is easy to see that N^{-1} is an R_1-ideal, and hence $N^{-1} \subset N$. Thus we have that $M \subsetneqq N^{-1} \subset N \subsetneqq R_1$. Since, by [9, Prop. 2.5], the length of R_1/M is 2, we must have that $N^{-1} = N$. It is easy to see that $N^{\#} = (MN)^{-1}$. Therefore,

$$R_2 = N^{\#} = (MN)^{-1} = (aN)^{-1} = N^{-1}a^{-1} = Na^{-1},$$

and hence $N = R_2 a$.

The remarks above concerning (b) and (c) have established the following statement.

(d) *There exists a chain of local rings* $R \subset R_1 \subset R_2 \subset \cdots \subset R_n \subset \cdots$ *such that each* R_i *is a reflexive ring;* R_{i+1} *is generated by two elements over* R_i *and hence is finitely generated over* R; *if* N_i *is the maximal ideal of* R_i, *then* R_{i+1} *is the dual of* N_i *with respect to* R_i; $N_i = R_{i+1} a$; *and* $M \subset N \subset N_2 \subset \cdots$. *This chain terminates if and only if* R_n *is a principal ideal ring, for some* n.

(e) R *is a Noetherian ring.*

Suppose that R is not a Noetherian ring. Then, according to Cohen's Theorem [10, Chapter I, Theorem 3.4], R has a nonfinitely generated prime ideal P. The ideal P is actually an R_n-ideal for each integer n. If R_n were a principal ideal ring, then P would be isomorphic to R_n and hence finitely generated over R. This contradiction shows that the chain of rings in (d) does not terminate.

Since each R_i is integral over R, there exists a chain

$$P \subset P_1 \subset P_2 \subset \cdots \subset P_n \subset \cdots,$$

where each P_i is a prime ideal of R_i, $P_{i+1} \cap R_i = P_i$, and $P_i \cap R = P$. It follows that $P_i \neq N_i$ for each i, and hence no power of a lies in any P_i. Let b be a nonzero element of P. Since $b \in M$, b/a is an element of R_1. We have that

$$\frac{b}{a} \cdot a = b \in P_1,$$

and hence $b/a \in P_1$. Therefore

$$\frac{b}{a^2} \in R_2 \quad \text{and} \quad \frac{b}{a^2} \cdot a^2 = b \in P_2,$$

which shows that $b/a^2 \in P_2$. By an obvious induction, we obtain that $b/a^n \in R_n$ for every n. But since P is an R_n-ideal, it follows that

$$\frac{b}{a^n} \cdot b = \frac{b^2}{a^n} \in P,$$

for every integer n. Hence

$$b^2 \in \bigcap_n Pa^n \subset \bigcap_k M^k = 0 .$$

This contradiction shows that R is a Noetherian ring.

(f) $\mu_*(R) \leq 2$

Since R is a Noetherian reflexive ring, it has Krull dimension 1, by [9, Theorem 3.8]. Thus if I is a nonzero ideal of R, R/I has finite length, which we shall denote by $\ell(R/I)$. Since $M = R_1 a$ is isomorphic to R_1, it is generated by two elements. Hence $\ell(M/M^2) = 2$, and thus $\ell(R/M^2) = 3$. We have seen in part (a) that

$$M^2 \subsetneqq Ra \subsetneqq M \subsetneqq R .$$

Hence $\ell(R/Ra) = 2$.

Let I be a nonprincipal ideal of R. From the exact sequence

$$0 \to \frac{Ra}{MI} \to \frac{R}{MI} \to \frac{R}{Ra} \to 0 ,$$

we conclude that $\ell(R/MI) = \ell(R/Ra) + \ell(Ra/MI)$. But $MI = R_1 aI = aI$, and hence $Ra/MI = Ra/Ia \cong R/I$. Thus we have that $\ell(R/MI) = 2 + \ell(R/I)$. From the exact sequence

$$0 \to I/MI \to R/MI \to R/I \to 0,$$

we conclude that $\ell(R/MI) = \ell(I/MI) + \ell(R/I)$. Putting the two equalities together, we have that $2 + \ell(R/I) = \ell(I/MI) + \ell(R/I)$. Therefore, $\ell(I/MI) = 2$, which shows that I can be generated by two elements. Thus we have proved the inequality $\mu_*(R) \leq 2$.

(3) ⇒ (1). The following argument is entirely due to Bass [2, Prop. 7.5], and we reproduce it here merely for the sake of completeness.

Let R be a Noetherian local domain with property FD. By the lemma, it is sufficient to prove that $\mu_*(R) \leq 2$. Let I be a nonzero ideal of R, and let $a_1, \cdots, a_n$ be a minimal generating set for I. We can assume that $n \geq 2$. Let $F = R^n$, $x = (a_1, \cdots, a_n) \in F$, and let B be the pure submodule of F generated by x. Then B is torsion-free of rank 1, and F/B is torsion-free of rank n - 1. By property FD, we have that $F/B = C_1 \oplus \cdots \oplus C_{n-1}$, where C_i is torsion-free of rank 1. By the theory of projective covers available to us over a local ring, there exists a direct-sum decomposition $F = F_1 \oplus \cdots \oplus F_{n-1}$ such that if $B_i = F_i \cap B$, then

$$B = B_1 \oplus \cdots \oplus B_{n-1} \quad \text{and} \quad C_i \cong F_i/B_i.$$

However, B is torsion-free of rank 1 and hence is indecomposable. Therefore, we can assume that $B \subset F_1$. Now the coordinates of x relative to any basis of F are a generating set for I. Hence, due to the fact that n is the minimal number of elements needed to generate I, B cannot be contained in any proper direct summand of F. Thus $F = F_1$, and n - 1 = 1. Hence n = 2, and $\mu_*(R) \leq 2$.

(2) ⇒ (3). Up to a point, our proof will be a modification of that of Bass [2, Prop. 7.2]. Since we have already proved that (2) implies (3), we can assume that R is a Noetherian local domain of Krull dimension 1 such that every ideal of R can be generated by two elements. Let A be a finitely generated, torsion-free R-module whose rank exceeds 1. We shall assume that A is indecomposable and obtain a contradiction.

Let $I = \sum_{f \in A^*} f(A)$ be the trace ideal of A.

Case I: I = R.

There exist elements $x_1, \cdots, x_n \in A$ and $f_1, \cdots, f_n \in A^*$ such that $\sum_{i=1}^{n} f_i(x_i) = 1$. Since R is a local ring, there is an integer i such that $f_i(x_i)$ is a unit of R. Hence there are elements $x \in A$ and $f \in A^*$ such that f(x) = 1. If we define $g: R \to A$ by g(r) = rx, then fg is equal to the identity on R. Hence Rx is a direct summand of A. This is a contradiction, and thus $I \neq R$.

Case II: $I \subset M$, *the maximal ideal of* R.

If M is projective, then, as we have seen in the proof of the implication (2) ⇒ (1), R is a principal ideal ring. But then A is a free R-module whose rank exceeds 1, and hence A cannot be indecomposable. Thus M is not projective, and $R_1 = M^{-1}$ is a ring. We shall show that A is an R_1-module.

Let $q \in R_1$, $x \in A$, and $f \in A^*$. We define (qx)(f) = qf(x). Since $M^{-1} \subset I^{-1}$ and $f(x) \in I$, it follows that $qf(x) \in R$. Thus $qx \in A^{**}$. Since R is a reflexive ring, we have that $A^{**} = A$, and hence $qx \in A$. Therefore, we have defined an operation of R_1 on A, extending that of R; in other words, A is an R_1-module.

Using the chain of rings $R \subset R_1 \subset R_2 \subset \cdots \subset R_n \subset \cdots$ established in part (d), we repeat our procedure and see that A is an R_n-module, for each n. If the chain terminates at R_n, then R_n is a principal ideal domain, and A decomposes. This contradiction shows that the chain does not terminate.

Let $S = \bigcup_{n=0}^{\infty} R_n$; then S is a ring, and S is not finitely generated as an R-module. Furthermore, A is an S-module by the preceding remarks. Let $x \neq 0 \in A$;

then $Sx \subset A$, and Sx is an R-submodule of A isomorphic to S. But every R-submodule of A is finitely generated. This contradiction shows that A is not indecomposable, and (3) is established.

This concludes the proof of the theorem.

COROLLARY 1. *Let* R *be a Noetherian integral domain. Then the following two conditions are equivalent:*

(1) *Every ideal of* R *can be generated by two elements.*

(2) R *is a Gorenstein ring of dimension* 1; *and if* M *is a maximal ideal of* R, *then either* M *is projective or* M^{-1} *is a Gorenstein ring of dimension* 1.

Proof. An examination of the proof of the implication (2) $\Rightarrow$ (1) of the theorem shows that the only place where we needed to know that every finitely generated ring extension of R is reflexive was in the proof that R was Noetherian.

COROLLARY 2. *Let* R *be a Noetherian domain such that every ideal of* R *can be generated by two elements. Let* I *be an ideal of* R. *Then there exists a finitely generated ring extension* R' *of* R *in* Q *such that* I *is a projective ideal of* R'. (*If* R *is local, then* I *is a principal ideal of* R'.)

Proof. Using the techniques of localization employed in the initial part of the theorem, we can assume without loss of generality that R is a local ring. Let

$$R \subset R_1 \subset R_2 \subset \cdots \subset R_n \subset \cdots$$

be the chain of rings established in part (d) of the theorem. If I is not a principal ideal of R_n for some n, then I is an ideal of $S = \bigcup_{n=1}^{\infty} R_n$; and the chain does not terminate. Let $x \neq 0 \in I$; then Sx is an ideal of R that is isomorphic to S. Since S is not finitely generated, this contradiction shows that I is a principal ideal of R_n, for some n.

Remarks. (1) One can show, using the techniques of the theorem, that a partial converse of Corollary 2 is true; namely that if R is a Noetherian local domain of Krull dimension 1 such that each ideal of R is a principal ideal of some finitely generated ring extension of R in Q, then the number of elements required to generate an ideal of R is bounded by the minimal number of generators of the maximal ideal of R. I do not know if, in fact, this number is two; that is, whether this is another characterization of the two-generator property.

(2) Let R be a Noetherian local domain such that every ideal of R can be generated by two elements. Let $S = \bigcup_{n=0}^{\infty} R_n$ be the ring established in part (d) of the theorem. Then S is a principal ideal domain, and if the chain does not terminate, then S is a discrete valuation ring. For if the chain terminates at n, then $S = R_n$ is a principal ideal domain, as we have seen. Hence suppose that the chain does not terminate. Then S is a local ring with principal maximal ideal Sa, as can readily be seen from part (a) of the theorem. It follows from Corollary 2 that every finitely generated ideal of S is principal. But this forces S to be a valuation ring. Thus $\bigcap_n Sa^n$ is a prime ideal of S whose intersection with R is zero, and hence $\bigcap_n Sa^n = 0$. It follows that S is a discrete valuation ring.

In all cases, S is integral over R and is integrally closed. Thus S is the integral closure of R. It now follows from [8, Cor. 7.5] that there is a decomposable factor module of Q if and only if the integral closure of R is a finitely generated principal ideal domain with exactly two primes.

(3) Let R be a Noetherian integral domain. It is easy to see that if the integral closure of R is a finitely generated R-module, or if R is a semilocal ring, then the conditions of the theorem are equivalent to the condition that R has property FD. It is an open question whether this is true in general; in other words, whether R has property FD if and only if R has property FD locally.

REFERENCES

1. H. Bass, *Torsion free and projective modules.* Trans. Amer. Math. Soc. 102 (1962), 319-327.

2. ———, *On the ubiquity of Gorenstein rings.* Math. Z. 82 (1963), 8-28.

3. H. Cartan and S. Eilenberg, *Homological algebra.* Princeton University Press, Princeton, N.J., 1956.

4. I. S. Cohen, *Commutative rings with restricted minimum condition.* Duke Math. J. 17 (1950), 27-42.

5. J. P. Jans, *Duality in Noetherian rings.* Proc. Amer. Math. Soc. 12 (1961), 829-835.

6. E. Matlis, *Some properties of Noetherian domains of dimension one.* Canad. J. Math. 13 (1961), 569-586.

7. ———, *Cotorsion modules.* Mem. Amer. Math. Soc. No. 49 (1964), 66 pp.

8. ———, *Decomposable modules.* Trans. Amer. Math. Soc. 125 (1966), 147-179.

9. ———, *Reflexive domains.* J. Algebra 8 (1968), 1-33.

10. M. Nagata, *Local rings.* Interscience Publ., New York, 1962.

11. O. Zariski and P. Samuel, *Commutative algebra,* Volume I. Van Nostrand, Princeton, N.J., 1958.

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