JOIN-PRINCIPAL ELEMENTS AND THE PRINCIPAL-IDEAL THEOREM

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In [2], R. P. Dilworth introduced the concept of a principal element of a multiplicative lattice and used it to isolate a class of multiplicative lattices in which many of the important theorems of classical ideal theory hold. He called these lattices Noether lattices and showed, among other things, that a Noether lattice satisfies the Noether decomposition theorems and the Krull Principal Ideal Theorem.

Since the concept of a Noether lattice is an abstraction of the lattice of ideals of a Noetherian ring, it is natural to ask when a Noether lattice can be represented as, or embedded in, the lattice of ideals of such a ring. Some results relating to this problem can be found in [1], [4], [5], and [6]. In particular, in [5], we proved that if $\mathscr{L}$ is a Noether lattice in which every maximal element is meet-principal, then every element of $\mathscr{L}$ is principal and $\mathscr{L}$ can be represented as the lattice of ideals of a ring. In this paper, we show that if 0 is prime in $\mathscr{L}$, then the same conclusion holds if every maximal element is join-principal (Theorem 2). This result is a consequence of Theorem 1, which generalizes the Principal-Ideal Theorem to elements that are either meet- or join-principal. Since, in general, the lattice of ideals of a Noetherian ring may have many elements that are join-principal but not principal, this extends the results of both Krull and Dilworth.

We use the notation and terminology of [2].

LEMMA 1. *Let* $\mathscr{L}$ *be a local Noether lattice in which* 0 *is prime. Let* $E \neq 0$ *be a join-principal element that is primary for the maximal element* M. *Then the rank of* M *does not exceed* 1.

Proof. Let d denote the rank of M. By the results of [3], there exists a polynomial p(x) of degree d - 1 such that, for all sufficiently large n, p(n) is the number of elements in a minimal representation of E as a join of principals. Let $E_1, \cdots, E_k$ be principal elements of $\mathscr{L}$ with join E, and let n be some positive integer. Then

$$E^{nk+n} = E^{nk}(E_1^n \vee \cdots \vee E_k^n),$$

and therefore $E^n = E_1^n \vee \cdots \vee E_k^n$, since E^{nk} is join-principal and 0 is prime. It follows that $p(n) \leq k$ for all sufficiently large n, and hence that $d \leq 1$.

THEOREM 1. *Let* $\mathscr{L}$ *be a local Noether lattice, and let* E *be an element of* $\mathscr{L}$ *that is either meet- or join-principal. Then the rank of every minimal prime of* E *does not exceed* 1.

Proof. Let P be a minimal prime of E. If E is meet-principal, then $\{E\}$ is meet-principal, and therefore it is join-irreducible in $\mathscr{L}_P$. It follows that $\{E\}$ is principal in $\mathscr{L}_P$ and therefore P has rank at most 1. On the other hand, if E is join-principal and P_0 is a second prime with $P_0 < P$, then $\{E \vee P_0\}$ is join-

Received August 18, 1969.

Michigan Math. J. 17 (1970).

principal and primary for the maximal element in $(\mathscr{L}/P_0)_P$. It follows that P has rank at most 1 in $\mathscr{L}/P_0$ and hence also in $\mathscr{L}$, since P_0 was arbitrary.

LEMMA 2. *Let $\mathscr{L}$ be a local Noether lattice in which* 0 *is a prime. If the maximal element* M *of $\mathscr{L}$ is join-principal, then* M *is principal.*

Proof. Let E be a principal element of $\mathscr{L}$ such that $E \leq M$ and $E \not\leq M^2$. By the previous theorem, the rank of M does not exceed 1; hence E is M-primary. Let n be the least positive integer such that $M^n \leq E$. If $n > 1$, then $M^n < E$. But in this case, $M^n \leq ME$, and therefore

$$M^{n-1} = M^n : M \leq ME : M = E .$$

Hence $n = 1$ and $M = E$.

THEOREM 2. *Let $\mathscr{L}$ be a Noether lattice in which* 0 *is prime and every maximal element is join-principal. Then every element is principal, and $\mathscr{L}$ can be represented as the lattice of ideals of a Noetherian ring.*

Proof. Let P be a maximal prime of $\mathscr{L}$. Since P is join-principal in $\mathscr{L}$, the prime $\{P\}$ is join-principal in $\mathscr{L}_P$. Hence $\{P\}$ is principal in $\mathscr{L}_P$, and the non-zero elements of $\mathscr{L}_P$ are precisely the powers of $\{P\}$ [5]. It follows [5] that every element of $\mathscr{L}$ is principal and that $\mathscr{L}$ can be represented as the lattice of ideals of a Noetherian ring.

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