A RELATION BETWEEN POINCARÉ DUALITY AND QUOTIENTS OF COHOMOLOGY MANIFOLDS

O. L. Costich

1. INTRODUCTION

A *proper map* is one for which the inverse image of each compact set is compact. Such a map is said to be *acyclic* over a coefficient domain L if each point-inverse is cohomologically trivial over L. Unless we say otherwise, we assume that L is an arbitrary but fixed principal ideal domain. We use the sheaf-theoretic cohomology theory and the homology theory defined by A. Borel and J. C. Moore as explicated in [3]. All supports for these theories are closed unless "c" appears as a subscript or superscript, in which case compact supports are to be taken. We replace "n-dimensional cohomology manifold" by the acronym "n-cm."

K. W. Kwun and F. Raymond [4] have proved the following result. Suppose that X is a compact, connected, orientable n-cm, and that Y is an n-cm. In addition, suppose that

$$f: (X, A) \to (Y, B) \qquad (A \neq X)$$

maps X - A onto Y - B and maps the closed set A onto B such that $f \mid X - A$ is acyclic. Then A satisfies a condition resembling Poincaré duality. More precisely, for $p \neq 0$ and $p \neq n$, the homomorphism

$$\phi: H_{n-p}(A) \xrightarrow{i_*} H_{n-p}(X) \xrightarrow{\Delta} H^p(X) \xrightarrow{i^*} H^p(A)$$

is an isomorphism, where i_*, i^* are induced by inclusion and Δ is the Poincaré duality isomorphism.

Theorem 1 of this paper provides a converse to the result of Kwun and Raymond in the case where B is a point and A is a continuum. If one assumes that f is proper and X is completely paracompact, the compactness of X may be discarded. Under these hypotheses, the assumption that A satisfies the homological condition above is sufficient to guarantee that Y is an orientable n-cm.

We apply Theorem 1 to give a generalized version of R. L. Wilder's monotone mapping theorem [5].

In what follows, X will denote a connected, orientable n-cm, and γ will denote the fundamental class of X ($\gamma \in H_n^c(X)$). If A is a continuum in X, then $c: X \to X/A$ is the canonical identification, and c(A) is represented by $*$. If $S \subset X/A$, then $c^{-1}(S) = S^*$. A proper, compact, connected subset A of X is called a *divisor* of X if the homomorphism

Received January 12, 1970.
This paper is part of the author's dissertation, prepared under the supervision of Professor K. W. Kwun at Michigan State University.

Michigan Math. J. 17 (1970).

$$\phi\colon H_{n-p}(A) \xrightarrow{i_*} H_{n-p}(X) \xrightarrow{(\gamma\cap)^{-1}} H^p(X) \xrightarrow{i^*} H^p(A)$$

is an isomorphism, for $p \neq 0$ and $p \neq n$.

2. STATEMENT AND PROOF OF THEOREM 1

THEOREM 1. *If* X *is a completely paracompact, orientable* n-cm *and* A *is a compact, connected subset of* X, *then* A *is a divisor of* X *if and only if* X/A *is an orientable* n-cm. *In either case, the sequences*

$$0 \longrightarrow H_q^c(A) \longrightarrow H_q^c(X) \longrightarrow H_q^c(X/A) \longrightarrow 0,$$

$$0 \longrightarrow H_c^q(X/A) \longrightarrow H_c^q(X) \longrightarrow H_c^q(A) \longrightarrow 0$$

are split-exact for $q \neq 0$.

Notice that if $\dim X = n$, then $\dim X/A = n$, where "dim" denotes "cohomological dimension."

LEMMA 1. *If* X *is an orientable* n-cm *and* A *is a compact, connected subset of* X, *then* A *is a divisor of* X *if and only if the homomorphism*

$$c_*\gamma\cap\colon H_c^{n-q}(X/A) \to H_q^c(X/A)$$

is an isomorphism.

Proof. Consider the commutative diagram, where $q \neq 0$ and $q \neq n$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_q^c(A) & \longrightarrow & H_q^c(X) & \longrightarrow & H_q^c(X/A) & \longrightarrow & 0 \\ & & \downarrow \phi & & \approx \uparrow \gamma\cap & & \uparrow c_*\gamma\cap & & \\ 0 & \longleftarrow & H_c^{n-q}(A) & \longleftarrow & H_c^{n-q}(X) & \longleftarrow & H_c^{n-q}(X/A) & \longleftarrow & 0. \end{array}$$

The rows are exact since $c\colon (X, A) \to (X/A, *)$ is a relative homeomorphism. A diagram chase shows that ϕ is an isomorphism if and only if $c_*\gamma\cap$ is an isomorphism. Moreover, in either case the rows are split.

LEMMA 2. *If* X *is an orientable* n-cm *and* A *is a divisor of* X, *then, for every open neighborhood* U *of* * *in* X/A *and for all* p, *the homomorphism*

$$\bar{c}^*\colon H_c^p(U) \to H_c^p(U^*)$$

induced by $c\colon X \to X/A$ *is a monomorphism, and for* $p = n$, *it is an isomorphism.*

Proof. Since the long exact sequence for cohomology is functorial, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_c^p(X/A) & \xrightarrow{c^*} & H_c^p(X) & \xrightarrow{h^*} & H_c^p(A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow r^* & & \downarrow = & & \\ \cdots & \longrightarrow & H_c^p(U) & \xrightarrow{\bar{c}^*} & H_c^p(U^*) & \xrightarrow{k^*} & H_c^p(A) & \longrightarrow & \cdots \end{array},$$

where $p > 0$ and the vertical maps are induced by inclusion, as are h^* and k^*. Since $k^* r^* = h^*$ is an epimorphism, k^* is an epimorphism (for $p > 0$). Thus $\bar{c}^*$ is a monomorphism for $p > 1$, and because $H_c^n(A) = 0$, $\bar{c}^*$ is an isomorphism for $p = n$. Because the reduced group $\tilde{H}^0(A)$ is trivial, $\bar{c}^*: H_c^1(U) \to H_c^1(U^*)$ is a monomorphism.

LEMMA 3. *If* X *is an orientable* n-cm *and* A *is a divisor of* X, *then the orientation sheaf* $\mathscr{H}_n(X/A)$ *is locally isomorphic to* L.

Proof. The lemma is obvious for points other than $*$. Let U and V be neighborhoods of $*$ in Y that are open and connected, with $U \subset V$. Since the dimension of U and V does not exceed n, the universal coefficient formula provides the commutative diagram

$$\begin{array}{ccc} H_n(V) & \xrightarrow{\approx} & \operatorname{Hom}(H_c^n(V), L) \\ r^*\big\downarrow & & \big\downarrow \operatorname{Hom}(j_{VU}, 1) \\ H_n(U) & \xrightarrow{\approx} & \operatorname{Hom}(H_c^n(U), L) \end{array},$$

where r_* is induced by restriction and j_{VU} by inclusion of cochains. But in the commutative diagram

$$\begin{array}{ccc} H_c^n(U) & \xrightarrow{\bar{c}^*} & H_c^n(U^*) \\ j_{VU}\big\downarrow & & \big\downarrow j_{V^*U^*} \\ H_c^n(V) & \xrightarrow{\bar{c}^*} & H_c^n(V^*) \end{array},$$

the $\bar{c}^*$ are isomorphisms, by Lemma 2. Moreover, $j_{V^*U^*}$ is an isomorphism [1]. Thus j_{VU} is an isomorphism; hence r_* is also an isomorphism. Because X is clc°, X/A is locally connected. The sheaf generated by

$$\{H_n(U)\colon U \text{ is a connected open neighborhood of } *\}$$

is isomorphic to the constant L-sheaf near $*$.

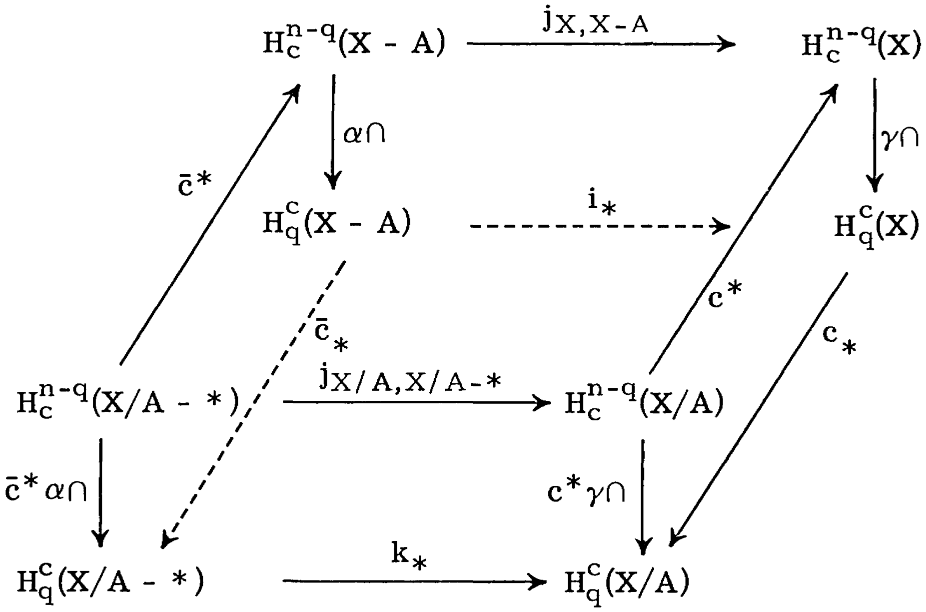
Recall that X is completely paracompact if every open subset of X is paracompact. This hypothesis will guarantee that Poincaré duality holds.

LEMMA 4. *If* X *is a completely paracompact, orientable* n-cm *and* A *is a divisor of* X, *then the local homology sheaves* $\mathscr{H}_q(X/A)$ *are* 0, *for* $q \neq n$.

Proof. The homology sheaf $\mathscr{H}_p(X/A)$ has stalks isomorphic to

$$H_p^c(X/A, X/A - y).$$

It thus suffices to show that $H_p^c(X/A, X/A - *) = 0$, for $p \neq n$. Consider the 3-dimensional diagram



,

where α is the fundamental class of $X - A$. The right and left faces commute, because the cap product is functorial, and the rear face commutes, because Poincaré duality commutes with inclusions. The commutativity of the top and bottom faces is guaranteed by the naturality of H_c^* and H_*^c for proper maps. Thus the front face also commutes. By the exact sequence of $(X/A, X/A - *)$, the map $j_{X,X-A}$ is an isomorphism for $q \neq 0$. Since $\bar{c}_*\alpha\cap = \bar{c}_* \circ \alpha\cap \circ \bar{c}^*$ is the composite of isomorphisms, it is an isomorphism, and $c_*\gamma\cap$ is an isomorphism, because A is a divisor. Thus k_* is an isomorphism for $q \neq n$. We conclude that

$$H_q^c(X/A, X/A - *) = 0,$$

for $q \neq n$.

LEMMA 5. *If* X *is an orientable* n-cm *and* A *is a divisor of* X, *then* X/A *is* clc.

Proof. Since X/A is finite-dimensional, it suffices to show that X/A is clc^∞. This amounts to proving that if U and W are open neighborhoods of $*$ that are relatively compact and satisfy the inclusion $\bar{U} \subset W$, then $\operatorname{im}[j_{WU}: H_c^p(U) \to H_c^p(W)]$ is finitely generated [3]. Because X is clc^∞, the condition $\overline{U^*} \subset W^*$ implies that $\operatorname{im}[j_{W^*U^*}: H_c^p(U^*) \to H_c^p(W^*)]$ is finitely generated. From the commutative diagram

$$\begin{array}{ccc} H_c^p(U) & \xrightarrow{\bar{c}^*} & H_c^p(U^*) \\ {\scriptstyle j_{WU}}\downarrow & & \downarrow{\scriptstyle j_{W^*U^*}} \\ H_c^p(W) & \xrightarrow{\bar{c}^*} & H_c^p(W^*) \end{array},$$

we see that $\operatorname{im}[\bar{c}^* \circ j_{WU}] = \operatorname{im}[j_{W^*U^*} \circ \bar{c}^*]$ is finitely generated. But $\bar{c}^*$ is a monomorphism; hence $\operatorname{im}[j_{WU}]$ is finitely generated.

Proof of Theorem 1. By Lemmas 3 and 4, if A is a divisor, then X/A is an orientable n-dimensional homology manifold. By Lemma 5, X/A is clc. According to [2], X/A is then an orientable n-cm.

3. STATEMENT AND PROOF OF THEOREM 2

THEOREM 2. *If* X *is a completely paracompact, orientable* n-cm *and* $f: X \to Y$ *is a continuous proper map onto a locally compact Hausdorff space such that each point-inverse is a divisor, then* Y *is an orientable* n-cm.

Proof. Let $y \in Y$, and let U and V be open, relatively compact neighborhoods of y such that $\overline{U} \subset V$. Since f is proper, $f^{-1}(U) = U^*$ and $f^{-1}(V) = V^*$ are relatively compact, open subspaces of X with $\overline{U}^* \subset V^*$. Because X is clc,

$$\mathrm{im}\,[j_{V^*W^*}: H_c^*(U^*) \to H_c^*(V^*)]$$

is finitely generated. Thus, by Poincaré duality, $\mathrm{im}\,[H_*^c(U^*) \to H_*^c(V^*)] = G(U^*, V^*)$ is finitely generated. Let $A_1, A_2, \cdots$ be point-inverses of $f \mid U^*$ that are nonacyclic. Since A_1 is a divisor of U^* and of V^*, and since the diagram

$$\begin{array}{ccc} & \nearrow & H_*^c(U^*) \\ H_*^c(A_1) & & \downarrow \\ & \searrow & H_*^c(V^*) \end{array}$$

is commutative, $H_*^c(A_1)$ is a proper direct summand of $G(U^*, V^*)$. In fact,

$$G(U^*, V^*) \approx H_*^c(A_1) \oplus G(U^*/A_1, V^*/A_1).$$

Repeating the process, we get the relation

$$G(U^*/A_1, V^*/A_1) \approx H_*^c(A_2) \oplus G[(U^*/A_1)/A_2, (V^*/A_1)/A_2].$$

Thus we obtain a strictly descending chain of submodules of $G(U^*, V^*)$. Such a chain must terminate with zero after a finite number of stages. Thus only finitely many point-inverses of $f \mid U^*$ can be nonacyclic. Using Theorem 1, we can shrink each nonacyclic point-inverse of f to a point and obtain an orientable n-cm Z and an acyclic map of Z onto Y. Wilder's monotone mapping theorem [5] applies and shows that Y is an orientable n-cm.

REFERENCES

1. A. Borel, *Seminar on transformation groups.* With contributions by G. Bredon, E. E. Floyd, D. Montgomery, R. Palais. Annals of Mathematics Studies, No. 46. Princeton Univ. Press, Princeton, N.J., 1960.

2. A. Borel and J. C. Moore, *Homology theory for locally compact spaces.* Michigan Math. J. 7 [1960], 137-159.

3. G. E. Bredon, *Sheaf theory.* McGraw-Hill, New York, 1967.

4. K. W. Kwun and F. Raymond, *Almost acyclic maps of manifolds*. Amer. J. Math. 86 (1964), 638-650.

5. R. L. Wilder, *Monotone mappings of manifolds*. Pacific J. Math. 7 (1957), 1519-1528. *Monotone mappings of manifolds*. II. Michigan Math. J. 5 (1958), 19-23.

Michigan State University
East Lansing, Michigan 48823
and
The University of Iowa
Iowa City, Iowa 52240