IMMERSIONS OF k-ORIENTABLE MANIFOLDS

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1. INTRODUCTION

Let M^m denote a smooth, closed, connected m-manifold. According to the classical theorems of Whitney, M^m embeds in R^{2m} and (if $m > 1$) immerses in R^{2m-1}. There are, however, many examples to show that the existence of an embedding $M^m \subset R^{2m-k+1}$ $(2 \leq k \leq m - 1)$ does not imply the existence of an immersion $M^m \subseteq R^{2m-k}$. In particular, complex projective space CP_m $(m = 2^r)$ embeds in R^{4m-1} [3] but does not immerse in R^{4m-2} [7]. In this note, we show that with additional restrictions, an embedding $M^m \subset R^{2m-k+1}$ will produce an immersion $M^m \subseteq R^{2m-k}$.

If α is a vector bundle over a CW-complex B, denote its stable equivalence class by (α). We say that (α) is *k-orientable* if the restriction of α to the k-skeleton of B is stably fibre-homotopy trivial. A manifold M^m (hereafter assumed to be smooth and connected) is k-orientable if its tangent bundle $\tau(M^m)$ is k-orientable. A map $f: M^m \to N^n$ between manifolds is *k-orientation-preserving* if $f^*(\tau(N^n)) - (\tau(M^m))$ is k-orientable. Let $i_0: N^n \to N^n \times R$ denote the inclusion $y \to (y, 0)$ $(y \in N^n)$. Our main result is the following.

THEOREM 1.1. *Suppose* $2k \leq m - 1$. *Let* M^m *be closed, and let*

$$f: M^m \to N^{2m-k}$$

be k-orientation-preserving. If the composition $i_0 f: M^m \to N^{2m-k} \times R$ *is homotopic to an embedding, then* f *is homotopic to an immersion.*

Some interesting corollaries follow.

COROLLARY 1.2. *Suppose* $2k \leq m - 1$. *Let* M^m *be closed and k-orientable. If* $M^m \subset R^{2m-k+1}$, *then* $M \subseteq R^{2m-k}$.

COROLLARY 1.3. *Suppose* $2k \leq m - 1$. *Let* $f: M^m \to N^{2m-k}$ *be given, where* M^m *is closed and* N^{2m-k} *is k-connected. Suppose either*

(a) M^m *is k-connected or*

(b) M^m *is* (k - 1)*-connected and* f *is k-orientation-preserving.*

Then f *is homotopic to an immersion.*

Proof. By A. Haefliger's embedding theorem [3], $i_0 f: M^m \to N^{2m-k} \times R$ is homotopic to an embedding. Now apply Theorem 1.1.

Note that, if M^m is (k - 1)-connected and $k \equiv 3, 5, 6,$ or $7 \pmod 8$, the assumption that f be k-orientation-preserving is superfluous. To verify this, let $\nu: M^m \to BO$ be a classifying map for $f^*(\tau(N^{2m-k})) - (\tau(M^m))$. There is a single obstruction to lifting ν to the k-connected covering BO[k] of BO. This occurs in

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$H^k(M^m; \pi_k(BO))$ and, by the Bott periodicity theorem, $\pi_k(BO) = 0$ if $k \equiv 3, 5, 6$, or $7 \pmod 8$.

COROLLARY 1.4. *Suppose* $2k \leq m - 1$. *Let* M^m *be closed,* $(k - 2)$*-connected, and* k*-orientable. Then* $M^m \subseteq R^{2m-k}$ *if and only if* $\overline{W}^{m-k+1} = 0$.

Proof. The necessity is clear. A. Haefliger and M. Hirsch [4] have shown that $M^m \subset R^{2m-k+1}$, provided $\overline{W}^{m-k+1} = 0$. The corollary now follows from Corollary 1.2.

Note that, if M^m is $(k - 2)$-connected and $k \equiv 6$ or $k \equiv 7 \pmod 8$, then M^m is automatically k-orientable.

J. Van Eps [9] has also obtained Corollary 1.4, by different techniques.

The type of argument used in Corollary 1.4 can be carried further. Suppose that M^m is k-orientable and r-connected ($k \leq 2r + 1$ and $2k \leq m - 1$). Let M_0^m denote M^m minus the interior of a smooth disk. If $M_0^m \subseteq R^{2m-k}$, then $M^m \subset R^{2m-k+1}$ [6]; now Corollary 1.2 implies that $M^m \subseteq R^{2m-k}$. That is, immersing M^m is equivalent to immersing M_0^m. The latter problem is more amendable to obstruction theory. In particular, using the work of E. Thomas [8], we can give cohomology conditions, involving secondary operations, for immersing a $(k - 3)$-connected manifold M^m in R^{2m-k}, for some values of m and k.

2. ORIENTABILITY

Let $\mathscr{S}$ denote the sphere spectrum whose nth term is S^n ($n \geq 0$), and let $\mathscr{S}^{(k)}$ denote the spectrum whose nth space $(S^n)^{(k)}$ is obtained by killing $\pi_r(S^n)$, for all integers $r \geq n + k$. We denote the natural map $\mathscr{S} \to \mathscr{S}^{(k)}$ by $\lambda^{(k)}$. If $\alpha = (E, B, p)$ is an orthogonal $(n - 1)$-sphere bundle, let

$$\Sigma(\alpha) = (\Sigma(E), B, \Sigma(p))$$

denote its fibrewise suspension, and let Δ_0 (respectively, Δ_1) denote the cross-section that sends $b \in B$ to the north (respectively, south) pole of $\Sigma(p^{-1}(b))$. The Thom space of α is $B^\alpha = \Sigma(E)/\Delta_0(B)$, and we regard $B \subset B^\alpha$ by the inclusion Δ_1.

Suppose h is a multiplicative cohomology theory. Recall that α is h-*orientable* if there exists an element $u \in \tilde{h}^n(B^\alpha)$ such that if $i_b: S^n \to B^\alpha$ identifies S^n with $\Sigma(p^{-1}(b))$, then $i_b^*(u)$ is a basis for the h(pt.)-module $\tilde{h}(S^n)$. If $\mathscr{F}$ is a ring spectrum (such as $\mathscr{S}$ or $\mathscr{S}^{(k)}$), we shall use the term $\mathscr{F}$-orientable rather than $h(\ ;\mathscr{F})$-orientable.

LEMMA 2.1. *A vector bundle* α *is* k*-orientable if and only if it is* $\mathscr{S}^{(k)}$*-orientable.*

Proof. The lemma is certainly true for $k = 1$. If $k > 1$, we may use the Thom isomorphism for singular cohomology to show that in the sequence

$$\tilde{h}^n(B^\alpha; \mathscr{S}^{(k)}) \xrightarrow{i^*} \tilde{h}^n((B^k)^{\alpha \mid B^k}; \mathscr{S}^{(k)}) \xleftarrow{\lambda_\#^{(k)}} h^n((B^k)^{\alpha \mid B^k}; \mathscr{S}),$$

both i^* and $\lambda_\#^{(k)}$ are onto. Here i denotes the inclusion. Therefore α is $\mathscr{S}^{(k)}$-orientable if and only if $\alpha \mid B^k$ is $\mathscr{S}$-orientable, and the latter holds if and only if $\alpha \mid B^k$ is stably fibre-homotopy trivial [1, Proposition (2.8)].

If $u \in \tilde{h}^n(B^\alpha; \mathscr{S}^{(k)})$ is a Thom class for α, the associated Euler class is $\chi = \Delta_1^*(u) \in h^n(B; \mathscr{S}^{(k)})$.

LEMMA 2.2. *Suppose* α *is* k-*orientable and* B *is* q-*coconnected* ($q \leq \min\{n+k, 2n-2\}$). *Then* α *admits a cross-section if and only if* $\chi = 0$.

Proof. Let f: $B \to BO$ be a classifying map for α, and let $\tilde{\chi} \in h^n(B, f; \mathscr{S})$ be an Euler class for α, as constructed in [2]. By Theorem (13.23) of [2], α has a cross-section if and only if $\tilde{\chi} = 0$. Since $q \leq n + k$, the map

$$\lambda_{\#}^{(k)}: h^n(B, f; \mathscr{S}) \to h^n(B, f; \mathscr{S}^{(k)})$$

is an isomorphism. Therefore $\tilde{\chi} = 0$ if and only if $\lambda_{\#}^{(k)}(\tilde{\chi}) = 0$. Finally, by [2, Lemma (13.20)], $\lambda_{\#}^{(k)}(\tilde{\chi}) = 0$ if and only if $\chi = 0$.

Our proof of Theorem 1.1 is based on the following observation, which may be of independent interest.

THEOREM 2.3. *Suppose* α *is* k-*orientable and* B *is* q-*coconnected* ($q \leq \min\{n+k, 2n-2\}$). *Then* α *admits a cross-section if and only if* B *is contractible in* B^α.

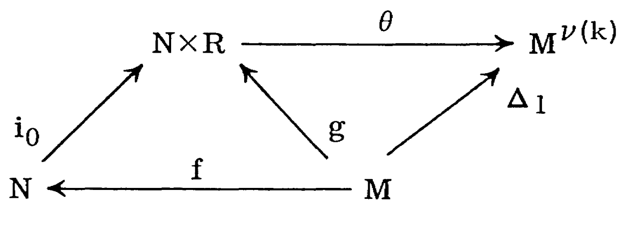
Proof. If δ: $B \to E$ is a cross-section, define a homotopy

$$D: B \times I \to \Sigma(E)$$

by $D(b, t) = [\delta(b), t]$ for $b \in B$ and $0 \leq t \leq 1$. Then D, followed by the collapsing map $\Sigma(E) \to B^\alpha$, is the desired contraction. On the other hand, if B is contractible in B^α, then Δ_1 is null-homotopic; hence $\chi = \Delta_1^*(u) = 0$. By Lemma 2.2, α has a cross-section.

3. PROOF OF THEOREM 1.1

Let g: $M \to N \times R$ be an embedding homotopic to $i_0 f$, and let $\nu(g)$ denote the normal bundle. We have the homotopy commutative diagram



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where θ is the Pontryagin-Thom map. By moving $N \times \{0\}$ up to a level $N \times \{t\}$, so that θ maps $N \times \{t\}$ to a point, we see that $\theta i_0 f$, and hence Δ_1, is homotopic to a constant. By Theorem 2.3, $\nu(g)$ has a cross-section; therefore $f^*(\tau(N)) - (\tau(M))$ has geometric dimension not exceeding $m - k$. By Hirsch's theorem [5], f is homotopic to an immersion.

Added March 9, 1970. It has come to my attention that D. Handel, in his paper *On the normal bundle of an embedding* (Topology 6 (1967), 65-68), has also proved Corollary 1.2 and has given other applications of this result.

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