NORMAL DECOMPOSITIONS IN SYSTEMS WITHOUT THE ASCENDING CHAIN CONDITION

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It is well known that if the family of ideals of a commutative ring satisfies the ascending chain condition (ACC), then every ideal has a finite primary representation [8]. This result has been generalized to various algebraic systems. In particular, the existence of primary representations was investigated in noncommutative rings by H. Tominaga [6]; in noncommutative rings that satisfy the ACC by J. A. Riley [5] and by W. E. Barnes and W. M. Cunnea [1]; and in lattices (with multiplication) that satisfy the ACC by M. Ward and R. P. Dilworth [7] and by R. P. Dilworth [2]. As in the case of noncommutative rings (D. C. Murdoch [4]), the ACC is not sufficient to guarantee the existence of primary representations in (commutative) multiplicative lattices [7]. It is, however, included in each set of sufficient lattice conditions given in [2] and [7]. In particular, Dilworth proved that if L is a modular, multiplicative lattice in which the ACC holds and in which each element is a join of meet-principal elements, then every element in L has a normal decomposition.

In this paper, we consider conditions weaker than the ascending chain condition in order to derive necessary and sufficient criteria for the existence of normal decompositions in multiplicative lattices. We do not use modularity, the existence of meet-principal elements, the ascending chain condition, or the condition that irreducible elements are primary. Since the lattice of ideals of a commutative ring (with an identity) is a multiplicative lattice, our results also extend the normal decomposition theory of Noetherian rings to a wider class of rings.

A *commutative, multiplicative lattice* is a complete lattice in which there is defined a commutative, associative, and join-distributive multiplication for which the greatest element, denoted by I, is the multiplicative identity. For basic properties of such lattices, see [2].

Throughout this paper, L will denote a commutative, multiplicative lattice. We say that L satisfies the *successive residual condition* if, for each element A in L, chains of the form

$$A:C_1 \leq A:(C_1 C_2) \leq \cdots \leq A:(C_1 C_2 \cdots C_n) \leq \cdots$$

are finite. This condition was used by W. Krull [3] to study minimal primes and associated primes of ideals. An element C ($C \neq I$) is the *residual component of* A *by* B (or simply a residual component of A) if there exists a positive integer n such that

$$C = A:B^n = A:B^{n+1} = \cdots .$$

We note that if A has a normal decomposition, then the residual components of A are precisely the isolated components of A. If each element A in L has only a finite number of distinct residual components, we say L satisfies the *residual-component*

Received February 3, 1969.

Michigan Math. J. 17 (1970).

condition. We say that L satisfies the RRC *condition* if for each A in L, the elements of the form $A:C$, where C is a residual component of A, satisfy the ascending chain condition. The lattice L satisfies the *radical condition* if every element of L contains a power of its radical. The BC *condition* holds in L if the equality

$$(A:B^n) \wedge (A \vee B^n) = A$$

holds for all A, B in L, for sufficiently large n. We say that Q *is primary for* P (or Q is P-primary) if $P^n \leq Q \leq P$ for some n and if the relation $AB \leq Q$ implies that $A \leq Q$ or $B \leq P$. An irredundant decomposition $A = Q_1 \wedge \cdots \wedge Q_k$ is a *normal decomposition* of A if Q_i is primary for, say, P_i and the P_i are distinct.

Together with the ascending chain condition, the BC condition plays a key role in the construction of normal decompositions in [1]. In this paper, we adapt the methods of [1] and prove that the following three statements are equivalent:

(A) Every element of L has a normal decomposition.

(B) L satisfies the residual-component condition, the BC condition, and the radical condition.

(C) L satisfies the BC condition, the successive residual condition, and the RRC condition.

H. Tominaga [6] gives a necessary and sufficient criterion (consisting of four conditions) for the existence of normal decompositions for two-sided ideals in general (noncommutative) rings. Although Tominaga's viewpoint and methods are considerably different from ours, his condition on residuals is directly related to our conditions. We reformulate his condition on residuals in a commutative multiplicative lattice L: For any elements D, E in L, the residual component of D by E exists, and L satisfies the residual-component condition. In our case, the existence of residual components follows from the BC condition. To see this, let D, E be elements in L, and choose n such that $D = D:E^n \wedge (D \vee E^n)$. From this we conclude that

$$D:E^n = D:E^{2n} \wedge (D \vee E^n):E^n = D:E^{2n}.$$

THEOREM 1. *If each element of* L *has a normal decomposition, then*

a) L *satisfies the residual-component condition,*

b) L *satisfies the* RRC *condition,*

c) L *satisfies the successive residual condition, and*

d) L *satisfies the* BC *condition.*

Proof. Let A be an element of L. Write

$$A = \bigwedge_{i=1}^{n} Q_i$$

(where Q_i is P_i-primary), and choose an integer k such that $P_i^k \leq Q_i$ for all i. For an element B, we have that

$$A:B^k = \bigwedge \{Q_i : B^k \mid B \not\leq P_i\} = \bigwedge \{Q_i \mid B \not\leq P_i\}.$$

Consequently, there are only finitely many residual components of A. The RRC condition is an immediate consequence of this.

To establish the successive residual condition, let

$$A : C_1 \leq A : (C_1 C_2) \leq \cdots \leq A : (C_1 C_2 \cdots C_m) \leq \cdots$$

be a chain of successive residuals of A. Assume the indexing in the above normal decomposition of A to be such that

$$\prod_{j=1}^{w_i} C_j \leq Q_i$$

(for some integer w_i) if and only if $i > h$. For each $i > h$, choose w_i to be the least such integer. There exists an integer s such that, for each integer t ($t \geq s$), the relation $C_t \not\leq P_i$ holds for all integers i ($i \leq h$). Let r be the maximum among s and the w_i ($i > h$). For an arbitrary integer q ($q \geq r$), let

$$B = \prod_{j=1}^{q} C_j \quad \text{and} \quad D = BC_{q+1},$$

and observe that $B \leq Q_i$ and $D \leq Q_i$ for all $i > h$. This implies that

$$A : B = \bigwedge_{i=1}^{h} (Q_i : B) \quad \text{and} \quad A : D = \bigwedge_{i=1}^{h} (Q_i : D).$$

Now choose a positive integer i such that $i \leq h$. The equality $Q_i : C_{q+1} = Q_i$ holds, since Q_i is P_i-primary and $C_{q+1} \not\leq P_i$. Consequently, the relations

$$Q_i : B = (Q_i : C_{q+1}) : B = Q_i : D$$

imply that $A : B = A : D$. Therefore, the successive residual condition is satisfied.

The BC condition is established essentially as in [1].

An element P is said to be a *maximal prime* of A if P is maximal in the set of elements B for which $A : B \neq A$. We note that if P is a maximal prime of A, then the relations

$$A : P = A : (A \vee P) \quad \text{and} \quad A : (P \vee XY) \leq A : ((P \vee X)(P \vee Y)) = (A : (P \vee X)) : (P \vee Y)$$

imply that $A \leq P$ and that P is prime.

The following lemma provides sufficient conditions for the existence of maximal primes.

LEMMA 1. *If* L *satisfies the* BC *condition, the residual-component condition, and the radical condition, or if* L *satisfies the successive residual condition and the* RCC *condition, then for each element* A *in* L ($A \neq I$), *there exists a maximal prime of* A.

Proof. Assume L satisfies the BC condition, the residual-component condition, and the radical condition. First, observe that the residual component of D by C exists (for all D, $C \in L$), by the BC condition. For an element A in L ($A \neq I$), let $P = \mathrm{Rad}(A:(A:B^k))$, where $A:B^k$ is a minimal residual component of A. Choose m so that $P^m \leq A:(A:B^k)$. The relations

$$A:B^k \neq A \quad \text{and} \quad A:B^k \leq A:(A:(A:B^k)) \leq A:P^m$$

imply that $A:P \neq A$. To see that P is a maximal prime of A, assume $P \leq R$ and $A:R \neq A$. Since $B^k \leq A:(A:B^k)$, we have $B \leq P \leq R$, so that

$$A:R^n \leq A:P^n \leq A:B^n = A:B^k$$

for all n. Hence, if $A:R^s$ is the residual component of A by R, then $A:R^s = A:B^k$, by the minimality of $A:B^k$. Therefore, $R^s \leq A:(A:R^s) \leq A:(A:B^k)$ and $R \leq P$.

Now, assume that L satisfies the successive residual condition and the RCC condition. For an element A in L ($A \neq I$), we define B to be a maximal element of the form $A:(A:S^k)$, where $A:S^k$ is a residual component of A. By the successive residual condition, there exists a residual component $Q = B:T^m$ of B such that Q has no proper residual components. Let $P = Q:C$ denote a maximal successive residual of Q ($P \neq I$), and let $Q:P^n$ be the residual component of Q by P. By our choice of Q, the residual component $Q:P^n$ is either Q or I. But the relation $Q:P^n = Q$ implies that

$$P = Q:C = (Q:P^n):C = (Q:C):P^n = I.$$

Since $P \neq I$, it follows that $Q:P^n = I$; hence $P^n \leq Q$. Since the relation $A:P = A$ implies that

$$Q = A:D = (A:P^n):D = (A:D):P^n = Q:P^n = I,$$

where $D = (A:S^k)T^m$, we conclude that $A:P \neq A$. Finally, to see that P is a maximal prime of A, assume that $P \leq R$ and $A:R \neq A$. Since $S^k \leq A:(A:S^k) \leq P$ and P is prime, we have $S \leq P \leq R$; hence $A:(A:S^m) \leq A:(A:R^m) \neq I$ ($m \geq k$), where $A:R^m$ is a residual component of A by R. By maximality of $A:(A:S^k)$, we conclude that $R^m \leq A:(A:R^m) = A:(A:S^m) \leq P$. Therefore $R = P$, and P is a maximal prime of A.

LEMMA 2. *Let* L *satisfy the* BC *condition, and let* A *be an arbitrary element in* L ($A \neq I$). *If* L *satisfies the residual-component condition and the radical condition, or if* L *satisfies the successive residual condition and the* RCC *condition, then for each maximal prime* P *of* A, *there exists a* P-*primary element* Q *such that* $A = (A:P^k) \wedge Q$, *for all* $k \geq 0$.

Proof. Assume that L satisfies the residual-component condition and the radical condition. Let P be a maximal prime of A, and choose n such that $A = (A:P^n) \wedge (A \vee P^n)$ and such that $A:P^n$ is a residual component of A by P. By the residual-component condition, there exists a residual component $Q = (A \vee P^n):E^t$ of $A \vee P^n$ that is maximal with respect to the property $A = (A:P^n) \wedge Q$ (since $A \vee P^n$ is a residual component of itself). Hence $A = (A:P^k) \wedge Q$ for all k.

To show that Q is the desired P-primary element, we prove first that P is a maximal prime of Q. Since P is a maximal prime of A and $A:P^n$ is a residual component of A by P, we have that

$$A \neq A:P = (A:P^n) \wedge (Q:P);$$

hence $Q: P \neq Q$. Let R be an arbitrary element such that $R \not\geq P$, and let $Q: R^m$ be a residual component of Q by R (hence $Q: R^m$ is a residual component of $A \vee P^n$ by R). The relation $R \not\geq P$ implies that $A: R = A$; hence

$$A = (A: P^n) \wedge (A: R^m) = (A: P^n) \wedge [((A: P^n) \wedge Q): R^m] = (A: P^n) \wedge (Q: R^m).$$

Consequently, $Q = Q: R$. Hence P is a maximal prime of Q.

To see that Q is P-primary, first observe that $P^n \leq Q$. Hence P is the radical of Q, and P is the unique maximal prime of Q. Finally, if $BC \leq Q$ and $C \not\leq P$, then $B \leq Q: C = Q$. Otherwise, we have $Q: (Q: C^r) \neq I$, where $Q: C^r$ is a residual component of Q by C. By the procedure outlined in the proof of Lemma 1, we can find a maximal prime P' of Q such that $P' \geq Q: (Q: C^r)$. Since P is the unique maximal prime of Q, we conclude that $P = P'$ and hence that $C \leq P$. Therefore, Q is P-primary.

A similar argument yields the result that if L satisfies the successive residual condition and the RCC condition, then for each maximal prime P of A, there exists a P-primary element Q such that $A = (A: P^k) \wedge Q$, for all $k \geq 0$.

THEOREM 2. *The following statements are equivalent:*

(A) *Every element of* L *has a normal decomposition.*

(B) L *satisfies the residual-component condition, the* BC *condition, and the radical condition.*

(C) L *satisfies the* BC *condition, the successive residual condition, and the* RRC *condition.*

Proof. In view of Theorem 1, we need only to prove that (B) implies (A) and (C) implies (A).

First we prove that (B) implies (A). Let A_1 be an element of L different from I, let P_1 be a maximal prime of A_1 (Lemma 1), and let Q_1 be a P_1-primary element such that $A_1 = (A_1 : P_1^n) \wedge Q_1$, for all n (Lemma 2). Choose n_1 such that

$$P_1^{n_1} \leq Q_1 \quad \text{and} \quad A_1 = (A_1 : P_1^{n_1}) \wedge (A \vee P_1^{n_1}).$$

In general, if A_s, Q_s, and n_s are defined such that

$$A_s = (A_s : P_s^{n_s}) \wedge Q_s = (A_s : P_s^{n_s}) \wedge (A \vee P_s^{n_s}) \quad \text{and} \quad P_s^{n_s} \leq Q_s,$$

we set $A_{s+1} = A_s : P_s^{n_s}$ and choose P_{s+1}, Q_{s+1}, and n_{s+1} for A_{s+1} as P_1, Q_1, and n_1 were chosen for A_1. Observe that whenever $i < j$, we have either $A_i = I$ or both $A_i \not\geq A_j$ and $P_i \not\leq P_j$. Since the elements A_i form a chain of residual components of A_1, we conclude that there exists a least integer $m + 1$ such that $A_{m+1} = I$. Therefore,

$$A_1 = \bigwedge_{i=1}^{m} Q_i$$

is a primary representation of A_1 in which all the primary components belong to distinct radicals (since $i < j$ implies $P_i \not\leq P_j$). Finally, to show that this primary representation is irredundant, suppose Q_i is redundant. Then the identities

$$A_1 = \bigwedge_{j \neq i} Q_j \quad \text{and} \quad A_i = A_{i-1} : P_{i-1}^{n_{i-1}} = A_1 : P_1^{n_1} P_2^{n_2} \cdots P_{i-1}^{n_{i-1}}$$

imply that $A_i = \bigwedge_{j > i} Q_j$. Consequently,

$$A_i : P_i = \bigwedge_{j > i} (Q_j : P_i) = A_i .$$

This contradiction completes the proof that (B) implies (A).

A similar argument shows that (C) implies (A).

Another property that has been studied in connection with primary decomposition is the Artin-Rees condition [5], [7]. A lattice L satisfies the Artin-Rees condition if for all A, B $\in$ L, the relation $A \wedge B^m \leq AB$ holds for some integer m. This simpler condition is equivalent to the BC condition if L is a modular lattice. Therefore, if L is modular, we can replace the BC condition by the Artin-Rees condition in Theorem 2.

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