ON TWO SUM THEOREMS FOR IDEALS OF C(X)

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1. INTRODUCTION

Let C(X) denote the ring of all continuous real-valued functions on a completely regular Hausdorff space X. It is well known (see [1, p. 198]) that in C(X) the sum of two z-ideals is a z-ideal and the sum of two prime ideals is a prime ideal.

L. Gillman and C. W. Kohls have remarked [2, p. 401] that the proofs of these assertions seem to depend strongly on properties of βX, the Stone-Čech compactification of X. The purpose of this note is to present elementary proofs of both theorems without using any properties of βX.

To emphasize that βX plays no apparent role, we prove the assertions for ideals of subrings of C(X), provided these subrings are absolutely convex sublattices of C(X). The methods of [1] do not seem to yield the sum theorem for z-ideals in such subrings.

2. PRELIMINARIES

An ideal I of a commutative ring R is said to be *semiprime* in R if, for each $x \in R$, we have that $x \in I$ whenever $x^2 \in I$. It is well known (see [1, p. 31]) and easy to prove that the semiprime ideals of a commutative ring are precisely the intersections of prime ideals.

A subring $\mathscr{A}$ of a lattice-ordered ring R is said to be *absolutely convex* in R if $x \in R$, $y \in \mathscr{A}$, and $|x| \leq |y|$ imply $x \in \mathscr{A}$. For the remainder of this note, let $\mathscr{A}$ denote some absolutely convex subring of C(X).

We remark that if f is an element of $\mathscr{A}$, then $|f|$ is an element of $\mathscr{A}$, since $|(|f|)| \leq |f|$.

We denote by Z(f) the set of all $x \in X$ such that $f(x) = 0$.

LEMMA 2.1. *A prime ideal* P *in* $\mathscr{A}$ *is absolutely convex in* $\mathscr{A}$.

Proof. Let $f \in \mathscr{A}$, $p \in P$, and suppose that $|f| \leq |p|$. Define g as in 5.5 of [1]; that is, let

$$g = \begin{cases} 0 & \text{on } Z(p), \\ \dfrac{f^2}{|p|} & \text{on } \sim Z(p). \end{cases}$$

Then g is easily seen to be continuous, and the inequality

$$|g| \leq \frac{|f| \cdot |f|}{|p|} \leq |f|$$

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implies that $g \in \mathscr{A}$. But $f^2 = g|p| \in P$ (since $|p| \in P$), and this implies that $f \in P$.

3. THE CONSTRUCTION

LEMMA 3.1. *Let* $f, a, b \in C(X)$, *and suppose* $Z(f) \supseteq Z(a) \cap Z(b)$. *Define*

$$h = \begin{cases} 0 & \text{on } Z(a) \cap Z(b), \\ \dfrac{fa^2}{a^2 + b^2} & \text{on } \sim[Z(a) \cap Z(b)] \end{cases}$$

and

$$k = \begin{cases} 0 & \text{on } Z(a) \cap Z(b), \\ \dfrac{fb^2}{a^2 + b^2} & \text{on } \sim[Z(a) \cap Z(b)]. \end{cases}$$

Then (i) $h, k \in C(X)$,

(ii) $|h| \leq |f|$ *and* $|k| \leq |f|$,

(iii) $f = h + k$,

(iv) $fa^2 = h(a^2 + b^2)$ *and* $fb^2 = k(a^2 + b^2)$.

Proof. Conclusions (ii), (iii), and (iv) are obvious. For (i), it suffices to show that h is continuous at each $x \in Z(a) \cap Z(b)$. Suppose $\varepsilon > 0$. Since $f(x) = 0$, there exists a neighborhood V of x such that $f(V) \subseteq (-\varepsilon, \varepsilon)$. By (ii), $h(V) \subseteq (-\varepsilon, \varepsilon)$, and therefore h is continuous at x.

4. THE SUM THEOREM FOR z-IDEALS

An ideal I of $\mathscr{A}$ is said to be a *z-ideal* (of $\mathscr{A}$) if $f \in \mathscr{A}$, $a \in I$, and $Z(f) \supseteq Z(a)$ imply $f \in I$.

THEOREM 4.1. *If* I *and* J *are z-ideals of* $\mathscr{A}$, *then* $I + J$ *is a z-ideal of* $\mathscr{A}$.

Proof. Choose $f \in \mathscr{A}$ with $Z(f) \supseteq Z(g)$ and $g \in I + J$, say $g = a + b$ for some $a \in I$, $b \in J$. Clearly, $Z(f) \supseteq Z(a) \cap Z(b)$. Let h and k be defined as in Lemma 3.1. Then $f = h + k$, and both h and k are members of $\mathscr{A}$, since $\mathscr{A}$ is absolutely convex. But $Z(h) \supseteq Z(a)$ implies $h \in I$, and $Z(k) \supseteq Z(b)$ implies $k \in J$; therefore $f \in I + J$.

5. THE SUM THEOREM FOR PRIME IDEALS

LEMMA 5.1. *If* I *and* J *are semiprime ideals of* $\mathscr{A}$, *then* $I + J$ *is semiprime in* $\mathscr{A}$.

Proof. Let $f^2 \in I + J$. Then $f^2 = a + b$ for some $a \in I$, $b \in J$. Again, since $Z(f) \supseteq Z(a) \cap Z(b)$, we may use the construction in Lemma 3.1 to obtain a decomposition $f = h + k$. Let P be prime in $\mathscr{A}$ with $P \supseteq I$. Then $h(a^2 + b^2) = fa^2 \in P$ implies $h \in P$ or $a^2 + b^2 \in P$. If $a^2 + b^2 \in P$, then $b \in P$ and hence $f \in P$. Since P is absolutely convex in $\mathscr{A}$, $|h| \leq |f|$ implies that $h \in P$. We have shown that

$$h \in \bigcap \{P \text{ prime} \mid P \supseteq I\},$$

and thus $h \in I$. Similarly $k \in J$, and therefore $f \in I + J$.

For the sake of completeness, we include the following theorem.

THEOREM 5.2. *Let* I, P, Q *be prime ideals of* $\mathscr{A}$, *and suppose* $I \subseteq P$ *and* $I \subseteq Q$. *Then either* $P \subseteq Q$ *or* $Q \subseteq P$.

Proof. Assume there exists $p \in P \setminus Q$. Then $p^2 \notin Q$, so that we may assume $p \geq 0$, without loss of generality. Now let $q \in Q$. Again we may assume q is non-negative. Define $f = p - q$ and consider $(f - |f|)(f + |f|) = 0$. Since I is prime, at least one of these factors must be in I. If $f + |f| \in I$, then $p - q + |f| \in Q$ implies that $p + |f| \in Q$. But Q is absolutely convex, and therefore this would imply that $p \in Q$, a contradiction. Thus $f - |f| \in I$. But $p - q - |f| \in P$ implies that $q + |f| \in P$ and hence $q \in P$.

We remark that Theorem 5.2 can be proved under considerably weaker hypotheses; namely, it is enough to assume that I is pseudoprime and P and Q are merely convex (see Section 4.1 in [2]). However, the methods employed in [1] and [2] to prove results of this type, while essentially the same as those used in Theorem 5.2, seem to be somewhat more complicated.

Now, instead of the theorem that the sum of two prime ideals is prime, we present the following slightly more general result.

THEOREM 5.3. *Let* I *and* J *be prime and semiprime, respectively, in* $\mathscr{A}$. *Then* $I + J$ *is prime in* $\mathscr{A}$.

Proof. Let $\mathscr{P} = \{P \mid P \text{ is prime in } \mathscr{A} \text{ and } P \supseteq I + J\}$. Then the members of $\mathscr{P}$ are linearly ordered by inclusion (each contains I), and hence it follows that $\bigcap \mathscr{P}$ is prime. But $I + J$ is semiprime by Lemma 5.1, so that $I + J = \bigcap \mathscr{P}$.

REFERENCES

1. L. Gillman and M. Jerison, *Rings of continuous functions*. Van Nostrand, Princeton, 1960.

2. L. Gillman and C. W. Kohls, *Convex and pseudoprime ideals in rings of continuous functions*. Math. Z. 72 (1959/60), 399-409.

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