THE K-NULLITY SPACES OF THE CURVATURE OPERATOR

Yeaton H. Clifton and Robert Maltz

1. INTRODUCTION

For any real constant K, set $K_{xy}(z) = R_{xy}(z) - K\{\langle x, z\rangle y - \langle y, z\rangle x\}$, where R denotes the curvature tensor, $\langle\ ,\ \rangle$ denotes the Riemannian inner product, and x, y, z belong to the tangent space M_m of the Riemannian manifold M, at the point m. Let $N_K(m) = \{x \in M_m \mid K_{xy} = 0 \text{ for all } y \in M_m\}$. We call $N_K(m)$ the K-*nullity space* at m, and we call $\mu_K(m) = \dim N_K(m)$ the *index of* K-*nullity* at m (T. Ôtsuki [6]).

$N_K(m)$ and $\mu_K(m)$ generalize the concepts of the *nullity space* $N_0(m)$ and of the *index of nullity* $\mu_0(m)$, which constitute the case K = 0. S. S. Chern and N. H. Kuiper [2] showed that N_0 defines an involutive distribution, and that if μ_0 is constant in a neighborhood, then the leaves of the resulting foliation are locally flat in the induced metric. R. Maltz [5] showed the following.

(i) The leaves are actually totally geodesic submanifolds of M (this implies they are locally flat).

(ii) If G denotes the open set on which μ_0 takes its minimum value m_0 (assumed to be positive), and if M is complete, then the leaves of the nullity foliation of G are also complete.

(iii) The nullity distribution N_0 has no isolated singular points (a singular point is a point at which the dimension μ_0 is not locally constant).

(iv) The boundary of G is the union of geodesics tangent to N_0.

Both involution of the distribution and property (i) are local, essentially algebraic results; since K_{xy} satisfies precisely the same algebraic conditions as R_{xy} (Ôtsuki [6]), it is obvious that N_K is involutive and has property (i) for all K (A. Gray [3]). It follows, of course, that the leaves of the foliation (for locally constant μ_K) have constant curvature K.

Properties (ii), (iii), and (iv), on the other hand, are global results. It is the purpose of this paper to establish them for arbitrary K. The essential idea is contained in the following result.

THEOREM (*). *Let* M *be a complete Riemannian manifold. Suppose* G *is an open subset of* M *on which the* K-*nullity index* μ_K *takes the constant value* m. *If* L *is a leaf of the* K-*nullity foliation induced on* G, *and if* $\gamma[0, c)$ *is a geodesic segment lying in* L, *then* $\lim_{t\to c^-}\gamma(t)$ *lies in* L *also.*

Remarks. (1) μ_K is easily seen to be upper-semicontinuous; therefore the set G on which μ_K attains its minimum value m_K is open. If $m_K > 0$, we actually obtain a foliation of G.

(2) It is easy to verify, by a simple generalization of Schur's Theorem, that no further generality can be obtained by allowing K to vary from point to point, except in the case where $\mu_K = 1$.

Received January 20, 1968.

Michigan Math. J. 17 (1970).

(3) If $\mu_K(m) \neq 0$ for some m and K, then $\mu_{K'}(m) = 0$ for all K' $(K \neq K')$; for if $x \in N_K(m)$ and $y \in N_{K'}(m)$ are orthogonal unit vectors, then $\langle R_{xy}(x), y \rangle = K = K'$, and therefore either N_K or $N_{K'}$ must be trivial.

(4) Manifolds for which $\mu_K > 0$ is constant might be called *quasi-constant-curvature manifolds,* since they constitute an obvious generalization of the constant-curvature case. Examples are provided by (a) product spaces of locally flat manifolds by arbitrary manifolds for K = 0, (b) spaces of recurrent curvature, where $\nabla R = R \otimes \alpha$ for some 1-form α (the components of the curvature tensor with respect to a parallel-frame field along a curve β are proportional to their initial values (Y. C. Wong [7], S. Kobayashi and K. Nomizu [4, p. 304]); therefore, if $p = \beta(0)$ and $q = \beta(1)$, it is easy to see that $\mu_K(p) = \mu_K(q)$; hence μ_K is constant on connected components). One verifies easily that Riemannian homogeneous spaces, symmetric spaces, and Lie groups have constant μ_K.

If μ_K is constant, then property (ii) is obvious, and (iii) and (iv) follow immediately.

2. GLOBAL PROPERTIES OF THE INTEGRAL MANIFOLDS

In the following, the symbol $\mathfrak{S}$ denotes cyclic summation.

LEMMA. *If* $[X, Y] = [X, Z] = [Y, Z]$ *for vector fields* X, Y, Z *on* M, *then* $\mathfrak{S}_{X,Y,Z} \nabla_X(R_{YZ}) = 0$. *Also* $\mathfrak{S}_{X,Y,Z} \nabla_X(K_{YZ}) = 0$.

Proof of the lemma. Bianchi's identity reads $\mathfrak{S}_{X,Y,Z}(\nabla_X R)_{YZ} = 0$. Expand

$$\nabla_X(R_{YZ}) = (\nabla_X R)_{YZ} + R_{\nabla_X Y, Z} + R_{Y, \nabla_X Z},$$

sum cyclically on X, Y, Z, and cancel, using the symmetry conditions on ∇. Then replace R by K everywhere.

Proof of Theorem (*). First, let $p = \gamma(0)$ and $\tilde{p} = \tilde{\gamma}(c)$, where $\tilde{\gamma}$ is the extension of γ in M; and let i, j, k $(1 \leq i, j, k \leq m)$ be nullity indices, α, β, γ $(m + 1 \leq \alpha, \beta, \gamma \leq d)$ nonnullity indices, and I, J, K $(1 \leq I, J, K \leq d)$ unrestricted indices.

Now we note that if $\xi = (x^1, \cdots, x^d)$ is a coordinate system in a neighborhood U of $\tilde{p}$, with $\partial/\partial x^1 = \gamma'$ along γ and with $\partial/\partial x^i$ nullity vector fields on $U \cap G$, then, by the lemma, $\mathfrak{S}\nabla_{\partial/\partial x^1}(K_{\partial/\partial x^\alpha\, \partial/\partial x^\beta}) = 0$.

It follows that $\nabla_{\partial/\partial x^1}(K_{\partial/\partial x^\alpha\, \partial/\partial x^\beta}) = 0$, since the second and third terms in the cyclic sum vanish identically in $U \cap G$, by nullity of $\partial/\partial x^1$. But this means that $K_{\partial/\partial x^\alpha\, \partial/\partial x^\beta}$ is parallel along γ in $U \cap G$. Now let $E = (E_1, \cdots, E_m, \cdots, E_d)$ be a parallel-frame field along $\tilde{\gamma}$: $[0, \infty) \to M$, the extension in M of γ to $[0, \infty)$ guaranteed by completeness. If $E_i(0) \in N_K(\gamma(0))$ and $E_\alpha(0) \in N_K^\perp(\gamma(0))$, then it follows from the total geodesity of L that E is adapted to N_K in G, in other words, that $E_i \in N_K$ and $E_\alpha \in N_K^\perp$ for all t. If $E_I(\tilde{p})$ is a nullity vector for some I, then $K_{\partial/\partial x^\alpha\, \partial/\partial x^\beta}(E_I)$ is a parallel-vector field along $\tilde{\gamma} \mid U \cap G$ vanishing at $\tilde{\gamma}(c)$, and therefore it must vanish identically by our assumption on $\tilde{\gamma} \mid U \cap G$. Hence $E_I \in N_K$ on $\tilde{\gamma} \mid U \cap G$. This proves that μ cannot increase at $\tilde{p}$.

We now establish the existence of a coordinate system ξ as above, starting with a Frobenius coordinate system $\eta = (y^1, \cdots, y^d)$ on a neighborhood V of $\gamma(0) = p$. We can further assume that $\eta(p) = (0, \cdots, 0)$, the origin in R^d, and that

$$(\partial/\partial y^1)_p = \gamma'(0), \qquad (\partial/\partial y^\alpha)_p \in N_K^\perp(p), \qquad \partial/\partial y^i \in N_K \quad \text{on } V.$$

(If η can be extended to a neighborhood of $\tilde{p}$, we can complete the proof as above; but in general this is impossible.)

Now let Σ be a slice of V determined by $y^i = 0$, and let

$$E = (E_1, \cdots, E_m, \cdots, E_d)$$

be a C^∞, orthonormal-frame field on Σ, adapted to the nullity distribution ($E_i \in N_K$), and such that $E_1(p) = \gamma'(0)$ (we can assume γ has unit speed). $\eta_2 = (y^{m+1}, \cdots, y^d)$ defines a coordinate system on Σ; set $\eta_2(\Sigma) = W \subset R^{d-m}$. Now define $F: R^m \times W \to M$ by

$$F(x^1, \cdots, x^m, \eta_2(s)) = \exp_s(\bar{x}),$$

where $s \in \Sigma$ and $\bar{x} = \sum x^i E_i(s)$. Since M is complete, F is defined for all values in R^m.

Identify $R^m \times W$ with a subset U of R^d, and let $u^1, \cdots, u^d$ be the natural Euclidean coordinate functions on U. Fixing $u^I = 0$ for all I ($I \neq 1$, $I \neq \alpha$) and restricting F to the plane so defined in U, we obtain an induced mapping $F_\alpha: R^2 \to M$, which is a rectangle in the sense of R. L. Bishop and R. J. Crittenden [1, p. 147]. Furthermore, the longitudinal curves of F_α are the geodesics $\exp_s(t E_1(s))$, where s is a point in the slice Σ_α of Σ defined by $u^\beta = 0$ for $\beta \neq \alpha$. It follows that the vector field X_α associated to F_α is a Jacobi vector field satisfying the Jacobi equation $X_\alpha'' = R_{X_\alpha \tilde{\gamma}'}(\tilde{\gamma}')$ along the geodesic $\tilde{\gamma} = \exp_p(t E_1(p))$, in particular. But

$$R_{X_\alpha \tilde{\gamma}'}(\tilde{\gamma}') = K\{\langle X_\alpha, \tilde{\gamma}'\rangle \tilde{\gamma}' - \langle \tilde{\gamma}', \tilde{\gamma}'\rangle X_\alpha\}$$

along γ, since $\gamma' \in N_K$. By Gauss's Lemma $\langle X_\alpha, \tilde{\gamma}'\rangle = 0$, since all longitudinal curves have unit speed, and $\langle X_2, \tilde{\gamma}'\rangle(0) = 0$ (since

$$X_\alpha(0) = dF_\alpha(\partial/\partial u^\alpha)_{\bar{0}} = (\partial/\partial y^\alpha)(p),$$

and the last member is assumed to be orthogonal to $N_K(p)$). Therefore $X_\alpha'' = KX_\alpha$. We have three cases:

$$K < 0 \text{ and } X_\alpha(t) = \sinh(\sqrt{-K}t) A_\alpha + \cosh(\sqrt{-K}t) B_\alpha,$$

$$K = 0 \text{ and } X_\alpha(t) = A_\alpha + tB_\alpha,$$

$$K > 0 \text{ and } X_\alpha(t) = \sin(\sqrt{K}t) A_\alpha + \cos(\sqrt{K}t) B_\alpha.$$

(Here A_α and B_α denote parallel vector fields along γ.)

In each case, we see that X_α is well defined, continuous, and bounded on $\tilde{\gamma}([0, c])$. (We are setting $X_\alpha(t) = X_\alpha(\gamma(t))$, of course.)

For $K \leq 0$, we show that F must be regular everywhere on $\tilde{\gamma}([0, c])$.

Let $X_\alpha^\perp(t)$ be the projection of $X_\alpha(t)$ onto the orthogonal complement $N_K^\perp(\gamma(t))$ of $N_K(\gamma(t))$, for $0 \leq t < c$. Define $X_\alpha^\perp(c) = \lim_{t \to c^-} X_\alpha^\perp(t)$. By continuity of N_K, we have $X_\alpha - X_\alpha^\perp \in N_K$ on $\tilde{\gamma}([0, c])$.

We now show that the $X_\alpha^\perp$ remain linearly independent on $\tilde{\gamma}([0, c])$. First of all, the $X_\alpha^\perp$ are linearly independent at p, since $X_\alpha(0) = dF(\partial/\partial u^\alpha)_p = (\partial/\partial y^\alpha)_p$ are assumed to be in $N_K^\perp(p)$. Hence $X_\alpha^\perp(0) = (\partial/\partial y^\alpha)_p$. Now suppose there is some linear combination $X = \sum c^\alpha X_\alpha^\perp$ such that $X(t_0) = \sum c^\alpha X_\alpha^\perp(t_0) = 0$ for some $t_0 \leq c$. Noting that

$$[X_\alpha, X_\beta] = dF([\partial/\partial u^\alpha, \partial/\partial u^\beta]) = 0,$$

$$[\gamma', X_\alpha] = dF([\partial/\partial u^1, \partial/\partial u^\alpha]) = 0, \quad \text{and } [\gamma', X_\beta] = 0,$$

we can apply the lemma again to find that $\mathfrak{S}\nabla_{\gamma'}(K_{X_\alpha} X_\beta) = \nabla_{\gamma'}(K_{X_\alpha} X_\beta) = 0$ along γ. Since K vanishes on the nullity components of X_α, we have $K_{X_\alpha^\perp} X_\beta = K_{X_\alpha} X_\beta$ on $\tilde{\gamma}([0, c])$. Hence it follows from $\nabla_{\gamma'}(K_{X_\alpha} X_\beta) = 0$ that the components of $K_{X_\alpha^\perp} X_\beta$ with respect to a parallel N_K-adapted frame field E(t) along γ are constants, and the same is true of the components of $K_{X X_\beta}$. But $K_{X X_\beta} = 0$ at t_0, since $X(t_0) = 0$. Hence $K_{X X_\beta} = 0$ everywhere on γ. In particular, this must be true at p and for all $\beta \geq m + 1$. But the X_β span $N_K^\perp$ at p, so that $K_{X X_\beta} = 0$ implies $X(0) \in N_K(p)$. On the other hand, $X(0) = \sum c^\alpha X_\alpha^\perp(0) \in N_K^\perp(p)$, which is possible only if $c^\alpha = 0$ for all α. Therefore the $X_\alpha^\perp$ must remain linearly independent on $\tilde{\gamma}([0, c])$.

Now define the map F_1 by $F_1(x^1, \cdots, x^m) = F(x^1, \cdots, x^m, 0, \cdots, 0)$. Then F_1 defines a regular mapping onto L for $K \leq 0$, because

$$F_1(x^1, \cdots, x^m) = \exp_p\left(\sum x^i E_i(p)\right) \in L,$$

and $d\exp_p$ is an isometry for $K = 0$ and is norm-increasing for $K < 0$. It follows immediately that F_1 is regular on the boundary of L as well, in particular at $\tilde{p}$. Hence the vectors $dF(\partial/\partial u^i) = dF_1(\partial/\partial u^i)$ are linearly independent at $\tilde{p}$. Furthermore, $dF(\partial/\partial u^i) \in N_K$ on L; hence $dF(\partial/\partial u^i)_{\tilde{p}} \in N_K(\tilde{p})$, by continuity.

Now we can see that F must be regular on $\tilde{\gamma}([0, c])$. First, let $\tilde{N}_K(t)$ be the m-plane at $\tilde{\gamma}(t)$ obtained by parallel translation of $N_K(0)$ along $\tilde{\gamma}$ ($N_K(t) = \tilde{N}_K(t)$ for $0 \leq t < c$). Then the $dF(\partial/\partial u^i)$ are linearly independent on $\tilde{\gamma}([0, c])$, and they span $\tilde{N}(t)$ $(0 \leq t \leq c)$. Furthermore, the $dF(\partial/\partial u^\alpha) = X_\alpha$ are linearly independent, and the $X_\alpha^\perp$ span $\tilde{N}^\perp(t)$ $(0 \leq t \leq c)$. Hence the rank of dF is exactly d everywhere on $\tilde{\gamma}([0, c])$.

In particular, F is regular at $\tilde{p} = \tilde{\gamma}(c)$; therefore F^{-1} defines a coordinate system $\xi = (x^1, \cdots, x^d)$ on a neighborhood U of F. Also, $\partial/\partial x^i \in N_K$ on $U \cap G$, and $\partial/\partial x^1 = \tilde{\gamma}'$ along $\tilde{\gamma}$. Hence, ξ is the required coordinate system, and the proof follows as in the first paragraph.

For $K > 0$, the map $F_1(x^1, \cdots, x^m) = \exp_p\left(\sum x^i E_i(p)\right)$ has critical points on the sphere of radius $\pi/\sqrt{K}$; but F is still regular on $\tilde{\gamma}([0, t))$, for $t < \pi/\sqrt{K}$. Therefore, if $c < \pi/\sqrt{K}$, then F^{-1} provides the required coordinate system. On the other hand, if $c > \pi/\sqrt{K}$, set $\tilde{\delta}(t) = \tilde{\gamma}(t - c + \pi/2\sqrt{K})$, and redefine F, using $\tilde{\delta}$ instead of $\tilde{\gamma}$ and $p' = \gamma(c - \pi/2\sqrt{K})$ instead of p. This completes the proof.

Property (ii) of Section 1 follows immediately from this lemma. The proofs of properties (iii) and (iv) follow exactly as in [5], and we refer the reader to that paper for the details.

REFERENCES

1. R. L. Bishop and R. J. Crittenden, *Geometry of manifolds*. Academic Press, New York, 1964.
2. S. S. Chern and N. H. Kuiper, *Some theorems on the isometric imbedding of compact Riemannian manifolds in euclidean space*. Ann. of Math. (2) 56 (1952), 422-430.
3. A. Gray, *Spaces of constancy of curvature operators*. Proc. Amer. Math. Soc. 17 (1966), 897-902.
4. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*. Interscience Publishers, New York, 1963.
5. R. Maltz, *The nullity spaces of the curvature operator*. Topologie et Géométrie Différentielle, Vol. 8, 20 pp. Centre Nat. Recherche Sci., Paris, 1966.
6. T. Ôtsuki, *Isometric imbedding of Riemann manifolds in a Riemann manifold*. J. Math. Soc. Japan 6 (1954), 221-234.
7. Y.-C. Wong, *Recurrent tensors on a linearly connected differentiable manifold*. Trans. Amer. Math. Soc. 99 (1961), 325-341.

University of California
Los Angeles, California 90024
and
University of California
Irvine, California 92664