INVARIANT SUBSPACES FOR ANALYTICALLY COMPACT OPERATORS

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Let T be a bounded operator on a Banach space, and let P be a nonzero polynomial. The theorem of A. R. Bernstein and A. Robinson [2] says that if P(T) is compact, then T has a nontrivial invariant subspace. An easy generalization is that if f is analytic in a neighborhood of the spectrum $\sigma(T)$ and f(T) is compact, while f is not identically zero, then T has a nontrivial invariant subspace. A related result of W. B. Arveson and J. Feldman [1] says that if a nonzero compact operator C is contained in the uniform closure $\mathscr{A}(T)$ of the polynomials in T, and if in addition T is quasi-nilpotent, then T has a nontrivial invariant subspace. (This was proved in [1] for operators on Hilbert space; several authors ([3], [4], [5]) have extended it to arbitrary normed spaces.) One might hope to generalize all these results by eliminating the hypothesis of quasi-nilpotence in the last theorem. Although we have not done so, we have produced another fragment of evidence in this direction, namely, the following.

THEOREM. *Let* T *be a bounded linear operator on a Banach space. Suppose that some analytic expression in* T *is a nonzero compact operator* C. *Then* T *has a nontrivial invariant subspace.*

Before we go into the proof, we should, of course, say what we mean by "analytic expression." To begin with, if f is analytic in a neighborhood of $\sigma(T)$, we call f(T) a *basic analytic expression* in T. Also, if $a_0, a_1, \cdots$ is a sequence of complex numbers such that $\sum a_n T^n$ converges in norm, we call this sum a *basic analytic expression.* By a *(general) analytic expression* we mean an element of the ring generated by the basic analytic expressions. A general analytic expression is thus an operator of the form $p(A_1, \cdots, A_n)$, where p is a polynomial and $A_1, \cdots, A_n$ are basic analytic expressions. (We could allow greater breadth to the concept of analytic expression, as will become clear in the course of our argument, but for the sake of simplicity we delimit it as indicated. We shall comment on this at the end of the paper.)

Note that all the analytic expressions belong to the uniformly closed algebra $\mathscr{B}(T)$ generated by T together with the operators $(\lambda - T)^{-1}$ $(\lambda \notin \sigma(T))$. The algebra $\mathscr{B}(T)$ is generally larger than $\mathscr{A}(T)$, although it coincides with $\mathscr{A}(T)$ if $\sigma(T)$ happens to be a single point.

We shall need two lemmas. The first of these is a generalization of Abel's theorem to series in a Banach space.

LEMMA 1. *Let* $A_0, A_1, A_2, \cdots$ *be elements of a Banach space. Suppose that the series* $\sum A_n$ *converges. Let* $0 < \theta < \pi/2$, *and let* S_θ *be the angular region*

$$S_\theta = \{z: |z| < 1, -\theta < \arg(1 - z) < \theta\}.$$

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Then

$$\lim_{\substack{z \to 1 \\ z \in S_\theta}} \sum_{n=0}^{\infty} A_n z^n = \sum_{n=0}^{\infty} A_n .$$

Proof. The proof mimicks the scalar case, but we give it for completeness. By replacing A_0 if necessary, we may assume that $\sum_{n=0}^{\infty} A_n = 0$. Define

$$B_n = A_0 + A_1 + \cdots + A_n, \qquad B_{-1} = 0 .$$

If $|z| < 1$, then

$$\sum_{n=0}^{\infty} A_n z^n = \sum_{n=0}^{\infty} (B_n - B_{n-1}) z^n = (1 - z)(B_0 + B_1 z + B_2 z^2 + \cdots)$$

$$= (1 - z)(B_0 + B_1 z + \cdots + B_n z^n) + R_n(z),$$

where

$$\|R_n(z)\| \leq |1 - z| \cdot \sup_{k > n} \|B_k\| \cdot (1 - z)^{-1} .$$

Now, if $z \in S_\theta$, it is easy to see that $|1 - z|/(1 - |z|)$ remains bounded as $z \to 1$. The lemma follows easily, since $B_k \to 0$. ∎

The next lemma establishes a boundedness property for the sum of a suitably convergent power series.

LEMMA 2. *Let* $S(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series convergent in an open disk* Δ. *Assume that in addition the series converges on a nontrivial arc* $\gamma \subset \partial\Delta$. *Then there exist a point* $a \in \gamma$ *and a number* $\delta > 0$ *such that* $S(z)$ *is bounded on the set* $\Delta \cap \{z: |z - a| < \delta\}$.

Proof. For notational convenience, we take Δ to have radius 1. If $p \in \partial\Delta$, define R_p to be the pie-shaped region

$$R_p = \{z: 0 < |p - z| < 1/2 \text{ and } -\pi/3 < \arg(1 - z/p) < \pi/3\} .$$

Then by Lemma 1, if $p \in \gamma$, then $\lim_{z \to p,\, z \in R_p} S(z)$ exists. Consequently $S(z)$ is bounded in R_p, for each $p \in \gamma$. In other words

$$\gamma = \bigcup_{n=1}^{\infty} E_n ,$$

where

$$E_n = \{p \in \gamma: |S(z)| \leq n \text{ for all } z \in R_p\} .$$

Moreover, each E_n is a closed set. Indeed, if $p \notin E_n$, then $|S(z)| > n$ for some $z \in R_p$; and if q is close enough to p, clearly $z \in R_q$ as well. Thus $\gamma \setminus E_n$ is open.

Now the Baire category theorem is applicable, and we conclude that some E_n contains an interior point a. Choose $\delta > 0$ so that $|p - a| < \delta$ implies $p \in E_n$. Then it is clear that $|S(z)| \leq n$ if $z \in \Delta$ and $|z - a| < \delta$. ■

COROLLARY 2.1. *By induction, the result above extends to the case of any finite number of power series* $S_1, \cdots, S_n$ *converging on* $\gamma \subset \partial\Delta$.

Proof of the theorem. Suppose that $\sigma(C) \neq \{0\}$. Then the compact operator C has some eigenvalue $\lambda_0 \neq 0$, and since T commutes with C, the eigenspace of C corresponding to λ_0 is a finite-dimensional subspace invariant under T.

Also, if $\sigma(T)$ is not connected, then it is the union $A \cup B$ of disjoint, nonvoid, closed sets. If γ is a smooth curve that encloses A and excludes B, then the operator

$$E = \frac{1}{2\pi i} \int_\gamma (\lambda I - T)^{-1} d\lambda$$

is a projection that commutes with T, and $0 \neq E \neq I$; therefore the range of E is a nontrivial invariant subspace for T.

Finally, if $\sigma(T)$ reduces to a single point, say λ_1, then C is a uniform limit of polynomials in the quasi-nilpotent operator $T - \lambda_1 I$; therefore, by [1], $T - \lambda_1 I$ has a nontrivial invariant subspace, which of course works for T as well.

Thus we may assume that $\sigma(T)$ is connected and has more than one point, while $\sigma(C) = \{0\}$. The remainder of our argument is devoted to showing that this case is, in fact, vacuous.

Let us assume that C is given by the expression

$$p(f_1(T), f_2(T), \cdots, f_m(T); S_1(T), S_2(T), \cdots, S_n(T)),$$

where $f_1, \cdots, f_m$ are analytic functions in a neighborhood U of $\sigma(T)$ and $S_1(T), \cdots, S_n(T)$ are norm-convergent series of the form

$$S_i(T) = \sum_{j=0}^{\infty} a_{ij} T^j.$$

Now, if $\lambda \in \sigma(T)$, then

$$|p(f_1(\lambda), \cdots, f_m(\lambda); \sum_{j=0}^{N} a_{1j}\lambda^j, \cdots, \sum_{j=0}^{N} a_{nj}\lambda^j)| \leq |\sigma(p(f_1(T), \cdots; \Sigma_0^N a_{ij} T^j, \cdots))|$$

$$\leq |\sigma(p(f_1(T), \cdots; \Sigma_0^N a_{ij} T^j, \cdots) - C)| + |\sigma(C)|,$$

where the last inequality is valid because the spectral radius is subadditive on commuting operators. But $|\sigma(C)| = 0$, while the spectral radius is never greater than the norm; therefore we have the inequality

$$|p(f_1(\lambda), \cdots; \Sigma_0^N a_{ij}\lambda^j, \cdots)| \leq \|p(f_1(T), \cdots; \Sigma_0^N a_{ij} T^j, \cdots) - C\|,$$

and the right-hand member converges to 0 as $N \to \infty$. That is,

$$\lim_{N\to\infty} p(f_1(\lambda), \cdots; \textstyle\sum_0^N a_{ij}\lambda^j, \cdots) = 0, \tag{1}$$

uniformly for $\lambda \in \sigma(T)$.

Let Δ be the open disk of radius $|\sigma(T)|$. Suppose that $\sigma(T) \cap \Delta \neq \emptyset$. Then, because $\sigma(T)$ is connected, it must contain an accumulation point in $\Delta \cap U$, where U is the neighborhood on which $f_1, f_2, \cdots$ are defined. Moreover, the series

$$S_i(\lambda) = \sum_{j=0}^{\infty} a_{ij}\lambda^j \qquad (1 \le i \le n)$$

converge on $\sigma(T)$ and therefore define analytic functions in Δ. Therefore

$$\Phi(\lambda) = p(f_1(\lambda), \cdots; S_1(\lambda), \cdots)$$

is analytic on $\Delta \cap U$. But by (1), Φ vanishes on $\sigma(T)$. Conclusion: Φ vanishes identically, because $\sigma(T)$ has an accumulation point in $\Delta \cap U$.

Furthermore, if $0 < r < 1$ and r is sufficiently close to 1, then

$$\sigma(rT) = r \cdot \sigma(T) \subset \Delta \cap U.$$

Hence

$$p(f_1(rT), \cdots; S_1(rT), \cdots) = \Phi(rT) = 0.$$

Now, as $r \to 1$, $f_i(rT) \to f_i(T)$ in norm, and by Lemma 1 it is also true that $S_i(rT) \to S_i(T)$. Consequently,

$$C = p(f_1(T), \cdots; S_1(T), \cdots) = \lim_{r\to 1^-} \Phi(rT) = 0,$$

which is contrary to the hypothesis.

The only remaining possibility is that $\sigma(T)$ is an arc γ on the boundary of Δ. But Corollary 2.1 then tells us that there exists an open set $V \subset \Delta$ such that $\partial V \cap \gamma$ is a nontrivial arc γ_1 and such that the functions $S_1(\lambda), \cdots, S_n(\lambda)$ are bounded on V. It follows that

$$\Phi(\lambda) = p(f_1(\lambda), \cdots; S_1(\lambda), \cdots)$$

is a bounded analytic function on V, and by (1) it has radial boundary value 0 on γ_1. Hence, by a well-known fact about H^∞-functions, we deduce that Φ vanishes in V and therefore in $U \cap \Delta$. Then, as before, we have the contradiction $C = 0$. ■

Remark. From the argument above we see that we can broaden the concept of "analytic expression" in T to include all operators of the form

$$C = \lim_{\substack{z\to 1\\ z\in S_\theta}} \Phi(zT),$$

where Φ is an analytic function defined on an open set W that contains $\sigma(zT)$ for all $z \in S_\theta$ sufficiently close to 1. Here θ can be any fixed positive angle.

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