R-AUTOMORPHISMS OF R[[X]]

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1. INTRODUCTION

In this paper, we assume that each ring is commutative and contains an identity element. If R is such a ring, then an endomorphism ϕ of R[[X]] is said to be an R-*endomorphism* if $\phi(r) = r$ for each element r in R. The results of this paper are closely related to those of M. J. O'Malley in [1] and of O'Malley and C. Wood in [2]. Before proceeding further, we summarize some of the results of [1] and [2] that are relevant to this paper.

Let S be a ring, and suppose that S contains an element b_0 such that S is a complete Hausdorff space in the (b_0)-adic topology. In [1, Theorem 2.1], O'Malley proved that for each element $\alpha = \sum_{i=0}^{\infty} a_i X^i$ of S[[X]] with $a_0 \in (b_0)$, there exists a unique R-endomorphism ψ_α of R[[X]] such that $\psi_\alpha(X) = \alpha$; moreover, ψ_α is onto if and only if a_1 is a unit of R, and if ψ_α is onto, ψ_α is also one-to-one. Conversely, if T is a ring, and if there exists a T-endomorphism f of T[[X]] such that

$$f(X) = \alpha = \sum_{i=0}^{\infty} a_i X^i,$$

where $\bigcap_{n=1}^{\infty} (a_0^n) = (0)$, then T is complete in the (a_0)-adic topology and $f = \psi_\alpha$ [1, Theorem 4.10]. The principal question that O'Malley leaves unanswered in [1] is the following:

(*) *Suppose that* R *is a ring. If there exists an* R-*automorphism of* R[[X]] *mapping* X *onto* $\sum_{i=0}^{\infty} a_i X^i$, *does it follow that* $\bigcap_{n=1}^{\infty} (a_0^n) = (0)$?

Corollary 5.5 of [1] shows that the answer to (*) is affirmative if

$$\bigcap_{n=1}^{\infty} (a_0^n) = a_0 \left[\bigcap_{n=1}^{\infty} (a_0^n) \right],$$

and hence the answer is affirmative if R is Noetherian or if a_0 is regular in R. Since a_0 must belong to the Jacobson radical of R [1, Lemma 5.1], O'Malley has characterized all R-automorphisms of R[[X]] when R is either a Noetherian ring, an integral domain, or a ring with the property that $\bigcap_{n=1}^{\infty} (a_0^n) = (0)$, for each a_0 in the Jacobson radical.

We shall refer to two results from [2]: If there exists an R-automorphism of R[[X]] mapping X onto $\sum_{i=0}^{\infty} b_i X^i$, then b_1 is a unit of R [2, Lemma 4.1]. As a consequence, it follows that if $\beta = \sum_{i=0}^{\infty} b_i X^i$ is an element of R[[X]], where b_1 is

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a unit of R, then there exists an R-automorphism of $R[[X]]$ mapping X onto β if and only if there exists an R-automorphism of $R[[X]]$ mapping X onto $b_0 - X$ [2, Lemma 4.2].

In Section 2, we prove that if the ring R contains an element a such that

$$a\left[\bigcap_{n=1}^{\infty} (a^n)\right] \subset \bigcap_{n=1}^{\infty} (a^n),$$

then the ring $S = R[[X]]/(a - X)$ is such that $S[[Y]]$ admits an S-endomorphism sending Y onto an element $\sum_0^\infty b_i Y^i$, where $\bigcap_{n=1}^\infty (b_0^n) \neq (0)$. We see thereby that the answer to (*) is negative in the general case. In Section 3, we give necessary and sufficient conditions, in the case of a general ring T, for $T[[X]]$ to admit a T-automorphism mapping X onto a prescribed element $\sum_0^\infty t_i X^i$ [Theorem 3.2]; further, when such a T-automorphism of $T[[X]]$ does exist, we are able to give conditions that are equivalent to the condition that $\bigcap_{n=1}^\infty (t_0^n) = (0)$ (Corollary 3.3).

2. A CONSTRUCTION USING POWER SERIES RINGS

In order to provide a negative answer to (*), we use the notion of the (R/A)-automorphism of $(R/A)[[X]]$ induced by a fixed R-automorphism of $R[[X]]$ that maps $A[[X]]$ onto itself. Our notation in Theorem 2.1 is as follows. By b_0 we denote a fixed element of R such that R is a complete Hausdorff space in the (b_0)-adic topology. Let $\beta = \sum_{i=0}^\infty b_i X^i \in R[[X]]$, and suppose that ϕ is the unique R-endomorphism of $R[[X]]$ mapping X onto β. Let A denote an ideal of R such that $\phi(A[[X]]) = A[[X]]$; hence ϕ induces an endomorphism of $R[[X]]/A[[X]]$. Since $R[[X]]/A[[X]] \simeq (R/A)[[X]]$, the endomorphism ϕ in turn induces an (R/A)-automorphism ϕ_A on $(R/A)[[X]]$. If we denote by r' the element $r + A$ of R/A, for each r in R, then ϕ_A satisfies the condition

$$\phi_A\left(\sum_{i=0}^{\infty} r_i' X^i\right) = \sum_{i=0}^{\infty} s_i' X^i,$$

where $\phi\left(\sum_0^\infty r_i X^i\right) = \sum_0^\infty s_i X^i$.

THEOREM 2.1. *If ϕ is onto, then ϕ_A is onto; if ϕ is one-to-one, then ϕ_A is one-to-one. The automorphism ϕ_A maps X onto $\sum_{i=0}^\infty b_i' X^i$, and $\bigcap_{n=1}^\infty (b_0')^n = (0)$ if and only if A is closed in the (b_0)-adic topology on* R.

Proof. The first sentence of Theorem 2.1 is clear. Under the one-to-one correspondence between ideals of R containing A and ideals of R/A, $A + (b_0^n)$ corresponds to $(b_0')^n$, so that $\bigcap_{n=1}^\infty [A + (b_0^n)]$ corresponds to $\bigcap_{n=1}^\infty (b_0')^n$. Hence $\bigcap_{n=1}^\infty (b_0')^n = (0)$ if and only if $\bigcap_{n=1}^\infty [A + (b_0^n)] = A$ — that is, if and only if A is closed under the (b_0)-adic topology.

To show that the answer to (*) is negative, it suffices, in view of Theorem 2.1, to find a ring R satisfying the following conditions. The ring R is a complete Hausdorff space in its (c_0)-adic topology, for some element $c_0 \in R$. Further, there exist an element $c = \sum_0^\infty c_i X^i \in R[[X]]$, where c_1 is a unit of R, and an ideal A of R such that $\psi_c(A[[X]]) = A[[X]]$, while A is not closed in the (c_0)-adic topology; here ψ_c denotes the unique R-automorphism of R[[X]] determined by the condition that $\psi_c(X) = c$.

At a glance, it might seem that the relation $\psi_c(A[[X]]) = A[[X]]$ holds for each ideal A of R and any such element c in R[[X]]. Indeed, this is true if A is finitely generated, because in this case,

$$\psi_c\left(\sum_1^k a_i R[[X]]\right) = \sum_1^k \psi_c(a_i R[[X]]) = \sum_1^k a_i R[[X]],$$

for any $a_1, a_2, \cdots, a_k$ in R. Hence, $\psi_c(A[[X]]) = A[[X]]$ if A is the intersection of a family of finitely generated ideals of R. If $c = c_0 - X$, then $\psi_c = \psi_c^{-1}$; hence the inclusion $\psi_c(A[[X]]) \subseteq A[[X]]$ implies that $\psi_c(A[[X]]) = A[[X]]$.

LEMMA 2.2. *Let* R *be a ring, choose* $a \in R$, *let* $A = \bigcap_{n=1}^\infty a^n R$, *and let* $\beta = \sum b_i X^i \in R[[X]]$, *where* $b_0 = a$. *Then* $A \subseteq (\beta)$ *if and only if* $aA = A$.

Proof. O'Malley proved in [1, Theorem 5.4] that if B is an ideal of R such that $aB = B$, then $B \subseteq \bigcap_{n=1}^\infty (\beta^n)$.

We prove, conversely, that the inclusion $A \subseteq (\beta)$ implies that $aA = A$. Thus each $t \in A$ has a representation

$$t = \left(\sum_0^\infty b_i X^i\right)\left(\sum_0^\infty r_i X^i\right),$$

for some element $\sum_0^\infty r_i X^i$ in R[[X]]. This equality gives rise to the following system of equations:

$$\begin{aligned} t &= b_0 r_0, \\ 0 &= b_0 r_1 + b_1 r_0, \\ &\vdots \\ 0 &= b_0 r_i + b_1 r_{i-1} + \cdots + b_i r_0, \\ &\vdots \end{aligned}$$

We prove that each r_i belongs to A. We prove first that each r_i belongs to (b_0). Thus $r_0 = -b_1^{-1} b_0 r_1 \in (b_0)$, and if $r_0, r_1, \cdots, r_s \in (b_0)$, then

$$r_{s+1} = -b_1^{-1}(r_0 b_{s+2} + \cdots + r_s b_2 + r_{s+2} b_0)$$

is also in (b_0). Further, if each r_i is in (b_0^k), then $r_0 = -b_1^{-1} b_0 r_1 \in (b_0^{k+1})$, and if $r_0, \cdots, r_s \in (b_0^{k+1})$, while r_t $(t > s)$ is in (b_0^k), then

$$r_{s+1} = -b_1^{-1}(r_0 b_{s+2} + \cdots + r_s b_2 + r_{s+2} b_0) \in (b_0^{k+1}).$$

It follows that each r_i is in A. In particular, $t = b_0 r_0 \in b_0 A = aA$. This implies that $A \subseteq aA$, and consequently, equality holds.

THEOREM 2.3. *Suppose that the ring* R *contains an element* a_0 *such that*

$$a_0 \left[\bigcap_{n=1}^{\infty} (a_0^n) \right] \subset \bigcap_{n=1}^{\infty} (a_0^n).$$

If $\alpha = \sum_{i=0}^{\infty} a_i X^i$, *where* a_1 *is a unit of* R, *then the ring* $S = R[[X]]/(\alpha)$ *has the property that* $S[[Y]]$ *admits an* S*-automorphism sending* Y *onto an element* $s_0 - Y$, *where* $\bigcap_{n=1}^{\infty} s_0^n S \neq (0)$.

Proof. The ring $R[[X]]$ is complete in the (X)-adic topology; hence there exists an $R[[X]]$-automorphism ϕ of $R[[X]][[Y]]$ sending Y onto $X - Y$. Since $A = \alpha R[[X]]$ is a finitely generated ideal of $R[[X]]$, ϕ maps $\alpha R[[X]][[Y]]$ onto itself. By Theorem 2.1, ϕ induces an S-automorphism ϕ_A of $S[[Y]]$ onto itself. To prove that the constant term of $\phi_A(Y)$ satisfies the condition in the theorem, we must show that $\alpha R[[X]]$ is not closed in the (X)-adic topology on $R[[X]]$. Since

$$\bigcap_{n=1}^{\infty} (a_0^n) \supset a_0 \left[\bigcap_{n=1}^{\infty} (a_0^n) \right],$$

Lemma 2.2 shows that $\bigcap_{n=1}^{\infty} (a_0^n) \not\subseteq \alpha R[[X]]$. We show, however, that the inclusion $\bigcap_{n=1}^{\infty} (a_0^n) \subseteq \bigcap_{n=1}^{\infty} (\alpha, X^n)$ holds, by proving that $a_0^n \in (\alpha, X^n)$, for each n. Thus $(\alpha, X^n) = (a_0 + a_1 X + \cdots + a_{n-1} X^{n-1}, X^n)$, and writing

$$g = -(a_1 + a_2 X + \cdots + a_{n-1} X^{n-2}),$$

we have the relation

$$a_0^n = a_0^n - X^n g^n + X^n g^n \in (a_0 - Xg, X^n) = (a_0 + a_1 X + \cdots + a_{n-1} X^{n-1}, X^n).$$

This completes the proof of Theorem 2.3.

To prove that the answer to (*) is negative, it therefore suffices to exhibit a ring R containing an element b such that $b\left[\bigcap_{n=1}^{\infty} (b^n) \right] \subset \bigcap_{n=1}^{\infty} (b^n)$. We present such a ring in Example 2.4. Although the construction used in Example 2.4 is quite intuitive, it seems that there should be an easier example than the one we give.

Example 2.4. Let S be a nonzero ring, and let $\{Y\} \cup \{X_i\}_{i=0}^{\infty}$ be a set of indeterminates over S. If $R = S[Y, \{X_i\}_0^{\infty}]/B$, where $B = (\{X_0 - X_i Y^i\})$, then the element $y = Y + B$ is such that $x_0 = X_0 + B \in \bigcap_{n=1}^{\infty} (y^n)$, while

$$x_0 \notin y \left[\bigcap_{n=1}^{\infty} (y^n) \right].$$

By passage to preimages under the natural homomorphism from $S[Y, \{X_i\}]$ onto R, we can verify the statements above by establishing the following three relations.

(1) $$\bigcap_{n=1}^{\infty} [B + (Y^n)] = (X_0, X_1 Y, X_2 Y^2, \cdots);$$

(2) $$B + Y\left\{\bigcap_{n=1}^{\infty} [B + (Y^n)]\right\} = (X_0 Y, X_0 - X_1 Y, X_0 - X_2 Y^2, \cdots);$$

(3) $$X_0 \notin (X_0 Y, X_0 - X_1 Y, \cdots, X_0 - X_n Y^n, \cdots).$$

While verification of these relations is rather detailed, it is nevertheless straightforward, and we omit the proof. The ring R provides the example we need, since $x_0 \in \left[\bigcap_{n=1}^{\infty} (y^n)\right] - \left[y\left\{\bigcap_{n=1}^{\infty} (y^n)\right\}\right]$.

3. EQUIVALENT CONDITIONS FOR THE EXISTENCE OF AN R-AUTOMORPHISM

In this section, we give necessary and sufficient conditions on a ring R in order that there exist an R-automorphism of R[[X]] mapping X onto a prescribed element β. Our conditions show that the construction in Example 2.4 is, in a sense, typical of the general case.

LEMMA 3.1. *Suppose that there exists an* R-*automorphism of* R[[X]] *mapping* X *onto* $\beta = \sum_{i=0}^{\infty} b_i X^i$. *Then, for each element* $\rho = \sum_{i=0}^{\infty} r_i X^i$, *where* r_1 *is a unit of* R *and where* r_0 *is a unit multiple of* b_0, *there exists an* R-*automorphism of* R[[X]] *mapping* X *onto* ρ.

Proof. Since there exists an R-automorphism of R[[X]] mapping X onto ρ if and only if there exists such an automorphism mapping X onto $r_0 - X$ [2, Lemma 4.2], we need to prove that there exists an R-automorphism of R[[X]] mapping X onto $r_0 - X$, under the assumption that there exists an R-automorphism ϕ mapping X onto $b_0 - X$. We let $r_0 = ub_0$, where u is a unit of R. Then there exists an R-automorphism τ of R[[X]] such that $\tau(X) = uX$. The R-automorphism $\tau^{-1}\phi\tau$ of R[[X]] has the property that

$$(\tau^{-1}\phi\tau)(X) = (\tau^{-1}\phi)(uX) = \tau^{-1}(ub_0 - uX) = ub_0 - X.$$

THEOREM 3.2. *Let* $\beta = \sum_0^{\infty} b_i X^i \in R[[X]]$. *The following three conditions are equivalent:*

1) *There exists an* R-*automorphism of* R[[X]] *that maps* X *onto* β.

2) *The mapping* $r \to r + (\beta)$ *of* R *into* $R[[X]]/(\beta)$ *is an isomorphism onto* $R[[X]]/(\beta)$.

3) $R[[X]] = R \oplus (\beta)$.

Proof. 1) → 2): Let ϕ be an R-automorphism of R[[X]] mapping X onto β, and let $\sigma = \phi^{-1}$; let μ be the homomorphism $f(X) \to f(0)$ of R[[X]] onto R. Then $\mu \circ \sigma$ is a homomorphism of R[[X]] onto R with kernel $\sigma^{-1}(\ker \mu) = \phi^{-1}(X) = (\beta)$. The induced isomorphism between R and $R[[X]]/(\beta)$ is defined by the relation $r \to r + (\beta)$.

2) $\rightarrow$ 3): Since the mapping $r \rightarrow r + (\beta)$ maps R onto $R[[X]]/(\beta)$, we have that $R[[X]] = R + (\beta)$; since $r \rightarrow r + (\beta)$ is one-to-one, it follows that $R + (\beta) = R \oplus (\beta)$.

It is clear that condition 3) implies condition 2).

2) $\rightarrow$ 1): We consider the $R[[X]]$-automorphism ϕ of $R[[X]][[Y]]$ that sends Y onto X - Y; this automorphism ϕ induces an $(R[[X]]/(\beta))$-automorphism ϕ_A of $(R[[X]]/(\beta))[[Y]]$. To be consistent with our notation in Section 2, we let $A = \beta R[[X]]$ in this case. We recall that ϕ_A is defined by the condition

$$\phi_A\left(\sum_0^\infty (r_i(X) + (\beta))Y^i\right) = \sum_0^\infty [s_i(X) + (\beta)]Y^i,$$

where $\phi\left(\sum_0^\infty r_i(X)Y^i\right) = \sum_0^\infty s_i(X)Y^i$.

The isomorphism $\mu: r \rightarrow r + (\beta)$ of R onto $R[[X]]/(\beta)$ and the automorphism ϕ_A give rise to an automorphism τ of $R[[Y]]$, where τ is defined by the relation $\tau\left(\sum_0^\infty r_i Y^i\right) = \sum_0^\infty \mu^{-1}(s_i(X) + (\beta))Y^i$, where $\phi\left(\sum_0^\infty r_i Y^i\right) = \sum_0^\infty s_i(X)Y^i$. In particular, $\tau(r_0) = r_0$ for each r_0 in R, and thus τ is an R-automorphism of $R[[Y]]$ that maps Y onto $\mu^{-1}(X) - Y$. Now $\mu^{-1}(X)$ is the unique element s of R such that $s - X \in (\beta)$. Thus we have that $R[[X]] = R \oplus (s - X)$, since 1) $\rightarrow$ 2) $\rightarrow$ 3), and we have that $R[[X]] = R \oplus (\beta)$, since 2) $\rightarrow$ 3). The inclusion $(s - X) \subseteq (\beta)$ implies that $(s - X) = (\beta)$, and $\beta = [s - X]u(X)$, for some unit $u(X) = \sum_0^\infty u_i X^i$ of $R[[X]]$. Thus $b_0 = su_0$, where u_0 is a unit of R, and $b_1 = su_1 - u_0$. Since s belongs to the Jacobson radical of R [1, Corollary 5.3], it follows that b_1 is a unit of R. By Lemma 3.1, we conclude that there exists an R-automorphism of $R[[X]]$ sending X onto β.

COROLLARY 3.3. *Suppose that there exists an* R-*automorphism of* $R[[X]]$ *mapping* X *onto* $\beta = \sum_{i=0}^\infty b_i X^i$. *The following conditions are equivalent.*

1) $\bigcap_{n=1}^\infty (b_0^n) = (0)$;

2) (β) *is closed in the* (b_0)-*adic topology on* $R[[X]]$;

3) (β) *is closed in the* (X)-*adic topology on* $R[[X]]$.

Proof. 1) $\leftrightarrow$ 2): By Theorem 3.2, the mapping $r \rightarrow r + (\beta)$ is an isomorphism of R onto $R[[X]]/(\beta)$. Under this mapping, (b_0^k) corresponds to $(b_0^k, \beta)/(\beta)$, and $\bigcap_{k=1}^\infty (b_0^k)$ corresponds to

$$\bigcap_{k=1}^\infty [(b_0^k, \beta)/(\beta)] = \left[\bigcap_{k=1}^\infty (b_0^k, \beta)\right]/(\beta).$$

Therefore, conditions 1) and 2) are equivalent.

2) $\leftrightarrow$ 3): We write $\beta = b_0 - X\alpha$, where $\alpha = -(b_1 + b_2 X + \cdots)$ is a unit of $R[[X]]$, since b_1 is a unit of R. Then $b_0^k \equiv X^k \alpha^k (\beta)$, so that

$$(\beta, b_0^k) = (\beta, X^k \alpha^k) = (\beta, X^k)$$

for each positive integer k. It follows that $\bigcap_{k=1}^\infty (\beta, b_0^k) = \bigcap_{k=1}^\infty (\beta, X^k)$; hence 2) and 3) are equivalent.

In Corollary 4.11 of [1], O'Malley establishes an equivalent form of the following result:

If there exists an R-*automorphism of* R[[X]] *mapping* X *onto* b_0 - X, *where* $\bigcap_{n=1}^{\infty} (b_0^n) = (0)$, *then, for each element* $r \in R$, *there exists an* R-*automorphism of* R[[X]] *mapping* X *onto* rb_0 - X.

In the general case, we are able to prove the preceding result only if r is in the Jacobson radical of R.

THEOREM 3.4. *If there exists an* R-*automorphism of* R[[X]] *mapping* X *onto* b_0 - X, *then, for each element* r *in the Jacobson radical of* R, *there exists an* R-*automorphism of* R[[X]] *that maps* X *onto* rb_0 - X.

Proof. We denote by ϕ the R-automorphism of R[[X]] that maps X onto b_0 - X. Since r is in the Jacobson radical of R, the element r - 1 is a unit of R. Thus, by Lemma 3.1 (or by Lemma 4.2 of [2]), there exists an R-automorphism σ of R[[X]] such that $\sigma(X) = b_0 + (r - 1)X$. Then

$$\phi\sigma(X) = b_0 + (r - 1)(b_0 - X) = rb_0 - (r - 1)X.$$

If μ is the R-automorphism of R[[X]] mapping X onto $(r - 1)^{-1} X$, it follows that $(\mu\phi\sigma)(X) = rb_0 - X$.

We conjecture that Theorem 3.4 is false if the hypothesis that r belongs to the Jacobson radical is dropped. To prove the conjecture, the element b_0, of course, must be such that $\bigcap_{n=1}^{\infty} (b_0^n) \neq (0)$, and herein lies our difficulty. The rings S of Theorem 2.3, which are such that S[[Y]] admits an S-automorphism mapping Y onto s_0 - Y, with $\bigcap_{n=1}^{\infty} (s_0)^n \neq (0)$, depend upon the existence of a ring R containing an element a such that

$$a\left[\bigcap_{n=1}^{\infty} (a^n)\right] \subset \bigcap_{n=1}^{\infty} (a^n).$$

The examples of such rings R of which we are aware are essentially modifications of Example 2.4, and these rings R discourage computations in $S = R[[X]]/(\alpha)$.

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