SINGULAR INNER FACTORS OF ANALYTIC FUNCTIONS

James G. Caughran and Allen L. Shields

If the analytic function f is in a Hardy class H^p $(p > 0)$, then f has a factorization $f = BSF$, where B is a Blaschke product, F is an outer function in H^p, and S, the *singular factor* of f, has the form

$$S(z) = \exp\left\{-\int \frac{e^{it}+z}{e^{it}-z}\, d\mu(e^{it})\right\},$$

where μ is a positive singular measure on the unit circle. The product BS is called the *inner factor* of f. See Chapter 5 of K. Hoffman's book [2] for a discussion.

If f is not constant, then the set of complex numbers c such that f - c has a nonconstant inner factor is uncountable; indeed, if c is in the range of f, then f - c has a nontrivial Blaschke factor. The situation is different if we restrict our attention to singular factors. We have the following result.

THEOREM 1. *If* $f' \in H^1$, *then* f - c *has a nonconstant singular factor for at most countably many values of* c.

The proof requires the following two results.

LEMMA 1. *Let* $\{\mu_\alpha\}$ *be a family of positive Borel measures on the unit circle with disjoint supports. If there exists a finite positive Borel measure* μ *such that* $\mu - \mu_\alpha \geq 0$ *for all* α, *then at most countably many of the measures* μ_α *are nonzero.*

The proof is easy. The proof of the next result appears in [1, Theorem 1] (see the note added in proof).

LEMMA 2. *If* $f' \in H^1$, *then the singular factor of* f *divides* f'.

Proof of Theorem 1. Since $f' \in H^1$, the function f is continuous on the closed unit disc (see [2, p. 70]). Let μ_c denote the singular measure associated with the singular factor of f - c. By a result of W. Rudin (see [4, Lemma 6]), f - c must vanish on the support of μ_c. Hence the measures $\{\mu_c\}$ have disjoint supports. But by Lemma 2, the singular factor of f - c must divide f', and hence it must divide the singular factor of f'. Thus, if μ is the singular measure associated with the singular part of f', then $\mu - \mu_c \geq 0$ for all c. The result now follows from Lemma 1.

If we drop the requirement that $f' \in H^1$ and require merely that $f \in H^\infty$, then the conclusion of the theorem need not hold.

THEOREM 2. *There exists an inner function* ϕ *such that* $\phi - c$ *has a nonconstant singular factor for uncountably many values of* c.

Proof. Let ϕ be some inner function that omits an uncountable set of values in the unit disc (for the existence of such functions see [3, Theorem 12 in Section II.4 and Footnote 3 on page 26]). Let c be an omitted value. Then

Received August 9, 1969.
This research was supported in part by the National Science Foundation.

$$\phi - c = \frac{\phi - c}{1 - \bar{c}\phi}(1 - \bar{c}\phi),$$

and the first factor on the right is an inner function (see [5, remarks following Theorem 2]), which must be singular since it has no zeros. The second factor is an outer function since its reciprocal is bounded. Thus $\phi - c$ has a nonconstant singular factor whenever c is an omitted value.

It would be interesting to narrow the gap between these two results. For example, does the conclusion of Theorem 1 remain valid if we require merely that f be continuous on the closed disc and that $f' \in H^p$ for some $p > 0$? This would be the case if we could prove Lemma 2 under this weaker hypothesis.

On the other hand, can we improve the counterexample in Theorem 2 so that $\phi' \in H^p$ for some $p > 0$ (or even, for all $p < 1$)? There would be hope of doing this if we could relate the size of the set of omitted values of a singular inner function to its associated singular measure. For example, if the measure is discrete (that is, if it has no continuous part), can the omitted set be uncountable? (The known examples of inner functions that omit prescribed sets of values arise from mapping the unit disc onto the universal covering surface of a disc from which a closed set of logarithmic capacity zero has been removed.) Even if progress is made in this direction, there will remain the question of finding conditions on the singular measure that guarantee that the derivative of the singular inner function is in H^p for some $p > 0$. It is easy to see that if the singular measure is discrete and if its masses are in ℓ^p for some $p < 1/2$, then the derivative is in H^p. However, we do not know an example of a singular inner function whose derivative is in $H^{1/2}$. Nor do we know an example of a singular inner function whose associated singular measure is continuous and whose derivative is in H^p for some $p > 0$.

REFERENCES

1. J. G. Caughran, *Factorization of analytic functions with* H^p *derivative.* Duke Math. J. 36 (1969), 153-158.

2. K. Hoffman, *Banach spaces of analytic functions.* Prentice-Hall, Englewood Cliffs, N. J., 1962.

3. K. Noshiro, *Cluster sets.* Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960.

4. W. Rudin, *The closed ideals in an algebra of analytic functions.* Canad. J. Math. 9 (1957), 426-434.

5. J. V. Ryff, *Subordinate* H^p *functions.* Duke Math. J. 33 (1966), 347-354.

The University of Kentucky
Lexington, Kentucky 40506
and
The University of Michigan
Ann Arbor, Michigan 48104