DISJOINTNESS OF MINIMAL SETS

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Let (X, T, π) and (Y, T, ρ) be two minimal transformation groups with compact Hausdorff phase spaces X and Y, respectively. The product of these two transformation groups, denoted by $(X \times Y, T, \omega)$, is a transformation group with the product space $X \times Y$ as phase space, with T as group, and with the action

$$\omega\colon ((x, y), t) = ((x, t)\pi, (y, t)\rho).$$

Generally, we shall not write out the action explicitly, that is, we shall write xt and ys for $(x, t)\pi$ and $(y, s)\rho$, when $x \in X$, $y \in Y$, $t, s \in T$. We say that two minimal transformation groups are *disjoint* if and only if the product transformation group of the two transformation groups is minimal. Let (Z, T) be a minimal transformation group. We say (Z, T) is a *factor of* (X, T) if there exists a continuous map $\theta\colon X \to Z$ such that $xt\theta = x\theta t$ for all $x \in X$ and $t \in T$. If (X, T) and (Y, T) are disjoint minimal transformation groups, they cannot have any nontrivial common factor [3], [4]. It is quite natural to consider the converse of the above statement (see [3, p. 1]). *If two minimal transformations* (X, T) *and* (Y, T) *have no nontrivial common factor, are they disjoint?* Under certain conditions, the answer to this question is positive. But in general, even if both minimal sets are distal, the answer is negative, as we shall show by examples.

Examples. Let X be the three-dimensional torus group; that is, let X be the product of three copies of the circle group $K = \{x \mid x \text{ a complex number}, |x| = 1\}$. Let

$$\exp \pi i\alpha, \quad \exp \pi i(2 - \alpha), \quad \exp \pi i\beta, \quad \exp \pi i(2 - \beta), \quad \exp \pi i\gamma$$

be rationally independent complex numbers, and define the homeomorphisms θ and ϕ on X by

$$\theta(x, y, z) = (x \exp \pi i\alpha,\ y \exp \pi i(2 - \alpha),\ xyz \exp \pi i\gamma),$$

$$\phi(x, y, z) = (x \exp \pi i\beta,\ y \exp \pi i(2 - \beta),\ xyz \exp \pi i\gamma).$$

Then θ and ϕ define two discrete flows on X, which we shall denote by (X, θ) and (X, ϕ), respectively. Both flows are distal and minimal. We shall prove this for (X, θ) only.

Let t_λ be a sequence of integers such that

$$\lim \theta^{t_\lambda}(x_1, y_1, z_1) = \lim \theta^{t_\lambda}(x_2, y_2, z_2).$$

Then

$$\lim x_1 \exp \pi i t_\lambda \alpha = \lim x_2 \exp \pi i t_\lambda \alpha,$$

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and *a fortiori*, $x_1 = x_2$. Similarly, $y_1 = y_2$. Thus

$$\lim z_1(x_1 y_1)^{t_\lambda} \exp \pi i t_\lambda \gamma = \lim z_2(x_2 y_2)^{t_\lambda} \exp \pi i t_\lambda \gamma,$$

$$\lim z_1 \exp \pi i t_\lambda \gamma = \lim z_2 \exp \pi i t_\lambda \gamma,$$

and therefore $z_1 = z_2$. Hence $(x_1, y_1, z_1) = (x_2, y_2, z_2)$, and (X, θ) is distal.

Now we show that (X, θ) is point-transitive. The orbit of $(1, 1, 1)$ is the set

$$\{(\exp \pi i n\alpha, \exp \pi i n(2 - \alpha), \exp \pi i n\gamma) \mid n \text{ an integer}\}.$$

It is the same as the discrete flow on X generated by the homeomorphism τ, defined by $\tau(x, y, z) = (x \exp \pi i\alpha, y \exp \pi i(2 - \alpha), z \exp \pi i\gamma)$. Since (X, τ) is minimal, *a fortiori* the orbit of $(1, 1, 1)$ in (X, τ) is dense in X. Thus, the orbit of $(1, 1, 1)$ in (X, θ) is dense in X. The flow (X, θ) is distal, and it is pointwise almost periodic. Hence (X, θ) is minimal.

It is easy to see that (X, θ) and (X, ϕ) are not almost-periodic. Actually, for any $x, y, z, z' \in K$, (x, y, z) and (x, y, z') are regionally proximal [2] in (X, θ).

Let Y be the two-dimensional torus group. Define the homeomorphisms $\bar{\theta}$ and $\bar{\phi}$ on Y by

$$\bar{\theta}(x, y) = (x \exp \pi i\alpha, y \exp \pi i(2 - \alpha)),$$

$$\bar{\phi}(x, y) = (x \exp \pi i\beta, y \exp \pi i(2 - \beta)).$$

The discrete flows $(Y, \bar{\theta})$ and $(Y, \bar{\phi})$ are almost-periodic. Since $\exp \pi i\alpha$, $\exp \pi i(2 - \alpha)$, $\exp \pi i\beta$, and $\exp \pi i(2 - \beta)$ are rationally independent, these two flows are disjoint.

It is easy to see that (X, θ) and (X, ϕ) have $(Y, \bar{\theta})$ and $(Y, \bar{\phi})$ as structure groups, respectively. Now, (X, θ) and (X, ϕ) cannot have nontrivial common factors. Suppose (Z, δ) is a nontrivial common factor of (X, θ) and (X, ϕ). The flow (Z, δ) is distal. It has a nontrivial structure group, and this structure group is the common factor of $(Y, \bar{\theta})$ and $(Y, \bar{\phi})$ [2]. We know that this is not possible, since $(Y, \bar{\theta})$ and $(Y, \bar{\phi})$ are disjoint. Hence (X, θ) and (X, ϕ) have no nontrivial common factor. However, (X, θ) and (X, ϕ) are not disjoint, because $X \times X$ is a six-dimensional torus. The orbit of $((1, 1, 1), (1, 1, 1))$ in $(X \times X, \theta \times \phi)$ is

$$\{(\exp \pi i n\alpha, \exp \pi i n(2 - \alpha), \exp \pi i n\gamma),$$
$$(\exp \pi i n\beta, \exp \pi i n(2 - \beta), \exp \pi i n\gamma) \mid n \text{ an integer}\},$$

and the orbit closure of the point is at most five-dimensional.

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