A CATEGORY SLIGHTLY LARGER THAN THE METRIC AND CW-CATEGORIES

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1. INTRODUCTION

A number of attempts have been made recently to construct a category suitable for algebraic topology. The class M_0 of metric spaces, despite its nice topological properties, is unsuitable because mapping cylinders cannot in general be formed in M_0, and M_0 does not contain all the CW-complexes. Both of these difficulties stem from the fact that M_0 is not closed under adjunction and weak union. In this paper we study the smallest category M that contains M_0 and is closed under adjunction and weak union. M-spaces can be constructed from metric spaces by a process analogous to the way CW-complexes are built up from cells. Many of the convenient separation properties of metric spaces, such as paracompactness, are shared by M-spaces.

Our work is related to that of Borges [1], Michael [10], and Steenrod [12]. In fact, M is a subcategory of Steenrod's CG-category; finite products and subspaces in M are exactly those of CG. In addition, every M-space is a stratifiable space of Borges, and every separable M-space is an $\aleph_0$-space of Michael.

One of the main reasons for introducing the category M is that it provides a natural setting for Hanner's generalization of Whitehead's extension of a theorem due to Borsuk. Hanner's result states roughly that a space obtained by adjoining an ANR(M_0) to an ANR(M_0) along an ANR(M_0) is itself an ANR(M_0), provided that it is metrizable. In M, this result holds without qualifications: a space obtained by adjoining an ANR(M) to an ANR(M) along an ANR(M) is itself an ANR(M). This is the main result of the last section of the paper.

After stating some preliminary definitions and results in Section 2, we define the category M in Section 3 and show that M is closed under adjunction and weak union in Section 4. In Section 5, we discuss the category CG of compactly generated spaces (k-spaces) and show that M is a subcategory of CG. We consider subspaces, product spaces, and function spaces in Sections 6 to 8. Section 9 deals with separable M-spaces and their relation to $\aleph_0$-spaces. We obtain some basic results in the theory of retracts in M in Section 10.

2. PRELIMINARIES

By a *space* we shall mean a topological space. A *pair* (Y, B) is a space Y together with a closed subset B. If (X, A) and (Y, B) are pairs such that $X \subset Y$ and $A = X \cap B$, then (X, A) is called a *subpair* of (Y, B). (Our definition of "subpair" is more restrictive than the usual definition, which requires only that $X \subset Y$ and $A \subset B$.) If, in addition, X is closed in Y, then (X, A) is a *closed subpair* of (Y, B). A *map* is a continuous function. All neighborhoods are open. We denote the interval [0, 1] by I.

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Of central importance in this paper are the notions of proclusion, adjunction, and weak union. We review them briefly in this section.

A surjection $p: X \to Y$ is called a *proclusion* (or *identification* or *quotient map*) if it has the property that $B \subset Y$ is closed if and only if $p^{-1}(B) \subset X$ is closed. $A \subset X$ is said to be *saturated* if $p^{-1}(p(A)) = A$. Consequently, if p is a proclusion, then $B \subset Y$ is closed (or open) if and only if it is the image of a saturated closed (or open) subset of X. Many results concerning proclusions can be found in the recent book of Dugundji [4, Chapter 6], from which we take the following:

PROPOSITION 2.1 (see [4, p. 124]). *Let* $p: X \to Y$ *be a proclusion, and let* $f: X \to Z$ *be a map. If* $fp^{-1}: Y \to Z$ *is single-valued, then it is continuous.*

COROLLARY 2.2. *If* $p: X \to Y$ *and* $q: X \to Z$ *are proclusions such that* $qp^{-1}: Y \to Z$ *and* $pq^{-1}: Z \to Y$ *are single-valued, then* qp^{-1} *and* pq^{-1} *are homeomorphisms.*

Given spaces X and Y, denote their topological sum by $X + Y$. More generally, given a collection of spaces $\{X_\alpha\}$, denote their topological sum by $+_\alpha X_\alpha$.

Suppose that (X, A) is a pair and that $f: A \to Y$ is a map. Let R be the equivalence relation on $X + Y$ generated by

$$a \sim f(a) \quad \text{for each } a \in A,$$

and let $X \cup_f Y$ be the set of equivalence classes of R with the (unique) topology such that the natural projection $p: X + Y \to X \cup_f Y$ is a proclusion. $X \cup_f Y$ is called the *adjunction space* obtained from X and Y under f. A category $\mathscr{C}$ is said to be *closed under adjunction* provided that $X \cup_f Y \in \mathscr{C}$ for each pair (X, A) with $X \in \mathscr{C}$, each $\mathscr{C}$-space Y, and each map $f: A \to Y$.

Suppose (X, A) is a pair and $f: A \to Y$ is a map. Let $q: X + Y \to X \cup_f Y$ be the natural projection. A map $p: X + Y \to Z$ is called an *adjunction map* for f if there exists a homeomorphism $h: X \cup_f Y \to Z$ such that the diagram

$$\begin{array}{ccc} X + Y & \xrightarrow{\;q\;} & X \cup_f Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle h} & \\ & Z & \end{array}$$

commutes. The quintuple (X, A, f, Y, p) is called a *presentation* for Z. Observe that p is a proclusion.

PROPOSITION 2.3 (see [4, p. 128]). *If* (X, A, f, Y, p) *is a presentation for* Z, *then* p *maps* Y *homeomorphically onto a closed subset of* Z, *and* p *maps* $X - A$ *homeomorphically onto an open subset of* Z.

PROPOSITION 2.4 [4, pp. 128-129]. *Let* (X, A) *be a closed subpair of a pair* (X_0, A_0), *and let* Y *be a closed subset of a space* Y_0. *Suppose that* $g: A_0 \to Y_0$ *is a map such that* $g(A) \subset Y$, *and let* $f: A \to Y$ *be the restriction of* g. *If*

$$p: X_0 + Y_0 \to X_0 \cup_g Y_0$$

is the natural projection, then $p(X + Y)$ *is closed in* $X_0 \cup_g Y_0$ *and* $p(X + Y)$ *is homeomorphic to* $X \cup_f Y$.

Let J^+ denote the set of nonnegative integers. Suppose that $\{X_n \mid n \in J^+\}$ is an increasing closed cover of a space X, in other words, that $X_n \subset X_{n+1}$ for all n, X_n is closed in X for all n, and $\bigcup_{n=0}^{\infty} X_n = X$. If the topology on X is such that a set $A \subset X$ is closed if and only if $A \cap X_n$ is closed for all n, then we say that X is the *weak union* of $\{X_n\}$ (in symbols: $X = \sum X_n$). A category $\mathscr{C}$ is said to be *closed under weak union* if $X \in \mathscr{C}$ whenever $X = \sum X_n$, where $X_n \in \mathscr{C}$ for all n.

Suppose that $\{X_n \mid n \in J^+\}$ is an increasing closed cover of a space X. For each n, let $g_n: X_n \to X$ be the inclusion. Then the collection $\{g_n\}$ defines a map $g: +_n X_n \to X$. The following two propositions are immediate consequences of the definition of weak union.

PROPOSITION 2.5. $X = \sum X_n$ *if and only if* g *is a proclusion.*

PROPOSITION 2.6. *If* $X = \sum X_n$, *then a function* $f: X \to Y$ *is continuous if and only if* $f \mid X_n$ *is continuous for all* n.

We shall also need the following result [12, Lemma 9.3].

PROPOSITION 2.7. *If* $X = \sum X_n$ *is a* T_1*-space and* C *is a compact subset of* X, *then there exists an index* n *such that* $C \subset X_n$.

Remarks. 1. Although we have defined weak union with J^+ as the index set, it will sometimes be notationally convenient to use the positive integers as the index set.

2. If $\{X_n \mid n \in J^+\}$ is a sequence of spaces such that X_n is a closed subset of X_{n+1} for all n, then we can topologize the set $X = \bigcup_n X_n$ by defining $A \subset X$ to be closed if and only if $A \cap X_n$ is closed in X_n for all n. With this topology, $X = \sum X_n$.

Before defining the category M, we formulate a definition of CW-complexes. Our approach will provide the motivation for the definition of M.

Definition 2.8. Let K be a space, and suppose that τ is a collection of pairwise disjoint open n-cells $(n \geq 0)$ such that $K = \bigcup_{t \in \tau} t$. For each $n \geq 0$, let $K^n = \bigcup \{t \in \tau \mid \dim t \leq n\}$. Suppose that

(1) K^0 is a discrete space (with topology inherited from K);

(2) for each $n > 0$ there exists a presentation $(X_n, A_n, f_n, Y_n, p_n)$ for K^n such that

(a) X_n is a free union of closed n-cells,

(b) A_n is the union of the boundaries of the cells in X_n,

(c) $p_n(Y_n) = K^{n-1}$; and

(3) $K = \sum K^n$;

then K (more properly, K together with τ) is called a CW-*complex*. τ is called a *triangulation* for K, and K^n is the n-*skeleton* of K.

This definition is equivalent to Spanier's [11, p. 401].

3. DEFINITION OF THE CATEGORY M

Having concluded with the preliminaries, we are now ready to define M-spaces and obtain some of their elementary properties.

Definition 3.1. A space Z is called an M-*space* if there exist subspaces $Z_0, Z_1, \cdots$ of Z such that

(1) Z_0 is metrizable;

(2) for each $n > 0$ there exists a presentation $(X_n, A_n, f_n, Y_n, p_n)$ for Z_n such that

(a) X_n is metrizable,

(b) $p_n(Y_n) = Z_{n-1}$; and

(3) $Z = \sum Z_n$.

The *category* M is the category of M-spaces and maps.

Observe that Definition 3.1 is obtained from Definition 2.8 simply by replacing cells and their boundaries by arbitrary metric spaces.

COROLLARY 3.2. *Every* CW-*complex is an* M-*space.*

Despite its simplicity, Definition 3.1 is difficult to use in practice, because the orderly stacked "skeleta" of 3.1(2) are difficult to manipulate. We need a somewhat cruder and more manageable stacking, and we therefore consider an alternate description of M-spaces.

Definition 3.3. A space is called an M_0-*space* if it is metrizable. Recursively, we define Z to be an M_{n+1}-*space* if there exists a presentation (X, A, f, Y, p) for Z such that X is metrizable and Y is an M_n-space. We shall refer to (X, A, f, Y, p) as an $(n+1)$-*presentation.* Finally, let $M_\infty = \bigcup_{n=0}^{\infty} M_n$.

Suppose that (X, A, f, Y, p) is a presentation for a space Z such that X is metrizable. We describe this situation by saying that "Z is obtained from p(Y) by adjoining a metric pair." With this terminology, condition 3.1(2) states that each Z_n $(n > 0)$ is obtained from Z_{n-1} by adjoining a metric pair, and condition 3.3 states that every M_{n+1}-space is obtained from an M_n-space by adjoining a metric pair. Recall that p(Y) is closed in Z and $p(Y) \cong Y$ (Proposition 2.3).

THEOREM 3.4. *A space* X *is an* M-*space if and only if there exist* M_∞-*spaces* $X_0, X_1, \cdots$ *such that* $X = \sum X_n$.

Proof. "Only if" is trivial. Conversely, suppose that $X_0, X_1, \cdots$ are M_∞-spaces such that $X = \sum X_n$. It follows at once from Definition 3.3 that for each $n \geq 0$ there exists a finite sequence of closed subspaces $X_{n,0}, X_{n,1}, \cdots, X_{n,m(n)}$ of X_n such that

(1) $X_{n,0}$ is metrizable,

(2) $X_{n,m(n)} = X_n$, and

(3) $X_{n,k+1}$ is obtained from $X_{n,k}$ by adjoining a metric pair $(0 \leq k < m(n))$.

Order the collection $\{X_{n,k} \mid 0 \leq n < \infty,\ 0 \leq k \leq m(n)\}$ by the rule $X_{n,k} < X_{n',k'}$ if either $n < n'$ or $n = n'$ and $k < k'$. With this ordering, the set $\{X_{n,k}\}$ is order-isomorphic to J^+. Let $Y_p = X_{n,k}$, where $p \leftrightarrow X_{n,k}$ under the order-isomorphism.

Define $Z_p = \bigcup_{j \leq p} Y_j$. Clearly, $\{Z_p | p \in J^+\}$ is an increasing closed cover of X. By (2), we see that $\{X_n | n \in J^+\}$ is a subcollection of $\{Z_p | p \in J^+\}$, cofinal with respect to inclusion, and it follows at once that

(4) $\sum Z_p = \sum X_n = X$.

We shall show that the collection $\{Z_p\}$ satisfies the conditions of Definition 3.1. Observe that $Z_0 = Y_0 = X_{0,0}$, and that $X_{0,0}$ is metrizable, by (1), so that condition (1) of Definition 3.1 is satisfied. Given $p \in J^+$, we shall show that Z_{p+1} can be obtained from Z_p by adjoining a metric pair. There are two cases:

Case I. Y_{p+1} is metrizable. Let $g: Y_{p+1} \cap Z_p \to Z_p$ be the inclusion. Then $Z_{p+1} \cong Y_{p+1} \cup_g Z_p$.

Case II. Y_{p+1} is not metrizable. Then, by (3), there exists a presentation (D, B, f, E, ψ) for Y_{p+1} such that D is metrizable and such that $\psi(E) = Y_p$. Let $D_0 = D \cap \psi^{-1}(Y_{p+1} \cap Z_p)$, and let $h = g\psi | D_0: D_0 \to Z_p$, where $g: Y_{p+1} \cap Z_p \to Z_p$ is the inclusion. Then $D \cup_h Z_p \cong Z_{p+1}$, since both are images of $D + E + Z_p$ under proclusions that satisfy the hypothesis of Corollary 2.2.

In either case, we obtain Z_{p+1} from Z_p by adjoining a metric pair. Therefore (2) of Definition 3.1 is satisfied, and (3) of Definition 3.1 follows from (4). This completes the proof.

The arbitrarily stacked M_∞-"skeleta" of Theorem 3.4 are more easily manipulated than the skeleta of Definition 3.1. We shall frequently use Theorem 3.4 without explicit reference. Observe that $M_0 \subset M_1 \subset \cdots \subset M_\infty \subset M$.

We close this section with some remarks concerning the separation properties of M-spaces.

Two of the most useful separation properties of metric spaces are paracompactness and perfect normality. (A space is perfectly normal if it is a normal Hausdorff space and all its closed sets are of type G_δ.) Both of these properties are implied by a stronger condition called *stratifiability* [1]. Since all metric spaces are stratifiable, and since stratifiability is closed under adjunction [1], it follows by an easy induction that all M_∞-spaces are stratifiable. Since stratifiability is closed under weak union (this is a special case of [1, Theorem 7.2]), we have the following result.

THEOREM 3.5. *All* M-*spaces are stratifiable; in particular, all* M-*spaces are paracompact and perfectly normal.*

COROLLARY 3.6. *All locally compact* M-*spaces are metrizable; in particular, all compact* M-*spaces are metrizable.*

Proof. All locally compact stratifiable spaces are metrizable [1].

4. CLOSURE OF M UNDER ADJUNCTION AND WEAK UNION

One serious defect of the metric category is its failure to be closed under adjunction and weak union. In this section, we shall prove that M is closed under these operations.

LEMMA 4.1. *Suppose that* (X, A) *is a pair and* $f: A \to Y$ *is a map. Suppose that* $X_0, Y_0, X_1, Y_1, \cdots$ *are spaces such that* $X = \sum X_n$ *and* $Y = \sum Y_n$. *Suppose also* *that* $f(A \cap X_n) \subset Y_n$ *for each* n. *If* $p: X + Y \to X \cup_f Y$ *is the natural projection, then* $X \cup_f Y = \sum p(X_n + Y_n)$.

Proof. By Proposition 2.4, $p(X_n + Y_n)$ is closed in $X \cup_f Y$ for all n. It follows readily that $\{p(X_n + Y_n) | n \in J^+\}$ is an increasing closed cover of $X \cup_f Y$. Suppose $A \subset X \cup_f Y$ meets $p(X_n + Y_n)$ in a closed set for all n. We must show that A is closed in $X \cup_f Y$. Since $A \cap p(X_n + Y_n)$ is closed, $p^{-1}(A) \cap X_n$ and $p^{-1}(A) \cap Y_n$ are closed in X_n and Y_n; and since $X = \sum X_n$ and $Y = \sum Y_n$, $p^{-1}(A)$ is closed in $X + Y$. But p is a proclusion; therefore, A is closed in $X \cup_f Y$.

LEMMA 4.2. *Let* (X, A) *be a pair, where* $X \in M_\infty$, *and let* $f: A \to Y$ *be a map. If* Y *is an* M-*space* (*or* M_∞-*space*), *then* $X \cup_f Y$ *is an* M-*space* (*or* M_∞-*space*).

Proof. We show first that if X is metrizable and Y is an M-space, then $X \cup_f Y$ is an M-space. Let d be a metric for X, and let $Y_0, Y_1, \cdots$ be M_∞-spaces such that $Y = \sum Y_n$. Without loss of generality, we may assume that $f(A) \cap Y_0 \neq \emptyset$. For each n, let $A_n = f^{-1}(Y_n)$; then $A_n \neq \emptyset$. Let

$$X_n = \begin{cases} \{x \in X | \ d(x, A_n) \leq n \cdot d(x, A - A_n)\} & \text{if } A \neq A_n, \\ X & \text{if } A = A_n. \end{cases}$$

The sequence $\{X_0, X_1, \cdots\}$ is an increasing closed cover of X. We shall now show that the sequence $\{\mathrm{int}(X_n) | n \in J^+\}$ is a cover of X, where $\mathrm{int}(X_n)$ denotes the interior of X_n in X. If $x \in X_n - A$, then by the definition of X_n, $x \in \mathrm{int}(X_{n+1})$. If $x \in A$, then there exist an $\varepsilon > 0$ and an $n > 0$ such that the open ε-ball (in A) centered at x lies in A_n — for otherwise there would exist a point x_n such that $d(x_n, x) < 1/n$ and $f(x_n) \notin Y_n$; since the set $\{x, x_1, x_2, \cdots\}$ is compact, the set $\{f(x), f(x_1), f(x_2), \cdots\}$ would be compact, in contradiction to Proposition 2.7. It follows that the open $\varepsilon/2$-ball (in X) centered at x lies in X_n; therefore $\{\mathrm{int}(X_n) | n \in J^+\}$ covers X.

We show next that $X = \sum X_n$. Suppose $Y \subset X$ meets each X_n in a closed set, and let x be a limit point of Y in X. Choose an index n such that $x \in \mathrm{int}(X_n)$; then x is a limit point of $Y \cap X_n$, and because $Y \cap X_n$ is a closed subset of X_n, $x \in Y$. Therefore Y is closed in X; hence $X = \sum X_n$.

For each n, let $f_n: A_n \to Y_n$ be the restriction of f. Since $X_n \in M_0$ and $Y_n \in M_\infty$, $X_n \cup_{f_n} Y_n \in M_\infty$, by Definition 3.3. By Proposition 2.4,

$$(X_n \cup_{f_n} Y_n) \cong \pi(X_n + Y_n),$$

where $\pi: X + Y \to X \cup_f Y$ is the natural projection, and, by Lemma 4.1,

$$X \cup_f Y = \sum \pi(X_n + Y_n);$$

therefore, $X \cup_f Y \in M$, by Theorem 3.4.

Suppose we have shown that the adjoining of any M_n-space to an M-space always yields an M-space, and suppose that X is an M_{n+1}-space. Let (Z, B, g, E, p) be an (n + 1)-presentation for X. Define a map $h: E \cap p^{-1}(A) \to Y$ by

$$h(x) = fp(x) \quad \text{for all } x \in E \cap p^{-1}(A).$$

By the induction hypothesis, $E \cup_h Y \in M$. Let $q: E + Y \to E \cup_h Y$ be the natural projection, and define a map $j: B \cup (Z \cap p^{-1}(A)) \to E \cup_h Y$ by

$$j(x) = \begin{cases} qg(x) & \text{if } x \in B, \\ qfp(x) & \text{if } x \in Z \cap p^{-1}(A). \end{cases}$$

The map j is well-defined; for, if x belongs to both B and $Z \cap p^{-1}(A)$, then

$$qfp(x) = qfpg(x) = qhg(x) = qg(x).$$

Since $Z \in M_0$ and $E \cup_h Y \in M$, it follows from the first part of this proof that $Z \cup_j (E \cup_h Y) \in M$. But $Z \cup_j (E \cup_h Y)$ and $X \cup_f Y$ are homeomorphic, since both spaces are images of $Z + E + Y$ under proclusions that satisfy the hypothesis of Corollary 2.2. Therefore, $X \cup_f Y \in M$.

Suppose now that $Y \in M_\infty$. If $X \in M_0$, then $X \cup_f Y \in M_\infty$, by Definition 3.3. Arguing as in the preceding paragraph, we see that $X \cup_f Y \in M_\infty$, where X is any M_∞-space. This completes the proof.

Remark. Suppose that X and Y are spaces such that $X \cap Y$ is closed in each. If we topologize $X \cup Y$ with the *union topology*, that is, if $A \subset X \cup Y$ is closed if and only if $A \cap X$ and $A \cap Y$ are closed in X and Y, respectively, then $X \cup Y \cong X \cup_i Y$, where $i: X \cap Y \to Y$ is the inclusion. Consequently, if X and Y are M_∞-spaces, then $X \cup Y$ is an M_∞-space, by Lemma 4.2.

LEMMA 4.3. *If* $X = \sum X_n$, *where* $X_0, X_1, \cdots$ *are* M-*spaces, then* X *is an* M-*space.*

Proof. For each n, let $X_{n0}, X_{n1}, \cdots$ be M_∞-spaces such that $X_n = \sum_m X_{nm}$. For each $k \in J^+$, let $Y_k = \bigcup_{n,m \leq k} X_{nm}$. If $A \subset X$ meets Y_k in a closed set for each k, then A meets X_{nm} in a closed set for all n and m. Since $X_n = \sum_m X_{nm}$ and $X = \sum X_n$, it follows that A is a closed subset of X; therefore, $X = \sum Y_k$. By the remark above, Y_k is an M_∞-space; therefore, X is an M-space, by Theorem 3.4.

LEMMA 4.4. *Suppose* (X, A) *is a pair and* $f: A \to Y$ *is a map. If* X *and* Y *are* M-*spaces, then* $X \cup_f Y$ *is an* M-*space.*

Proof. Let $X_0, X_1, \cdots$ be M_∞-spaces such that $X = \sum X_n$. For each n, let $f_n: A \cap X_n \to Y$ be the restriction of f. Taking $Y_n = Y$ in Lemma 4.1, we see that $X \cup_f Y = \sum p(X_n + Y)$, where $p: X + Y \to X \cup_f Y$ is the natural projection. By Proposition 2.4, $p(X_n + Y) \cong X_n \cup_{f_n} Y$, and, by Lemma 4.2, $X_n \cup_{f_n} Y$ is an M-space; the result now follows from Lemma 4.3.

COROLLARY 4.5. *If* X *and* Y *are* M-*spaces and* $X \cap Y$ *is closed in each, then* $X \cup Y$, *under the union topology, is an* M-*space.*

Combining Lemmas 4.3 and 4.4 and Definition 3.1, we have the main result of this section.

THEOREM 4.6. *The category* M

(1) *is a full subcategory of the topological category,*

(2) *is closed under adjunction and weak union, and*

(3) *contains* M_0;

if M' *is any category possessing properties* (1) *to* (3), *then* M' *contains* M.

5. COMPACTLY GENERATED SPACES

A Hausdorff space X is said to be *compactly generated*, or to be a k-*space*, if it has the property that a set $A \subset X$ is closed if it meets every compact set in a closed set. The category CG of compactly generated spaces has been studied in [12].

Let τ be a Hausdorff topology on a set X. Define a topology $k\tau$ on X by defining a set to be closed if it meets each compact subset of (X, τ) in a closed set. The assignment $(X, \tau) \to (X, k\tau)$ defines a functor (in fact, a retraction) k from the category of Hausdorff spaces and maps onto CG [12].

PROPOSITION 5.1 [4, p. 248]. *If* X *is compactly generated and* Y *is a Hausdorff space, and if* $f: X \to Y$ *is a proclusion, then* Y *is compactly generated.*

THEOREM 5.2. *Every* M-*space is compactly generated.*

Proof. Since the natural projection in an adjunction is a proclusion, and since the composite of two proclusions is a proclusion, it follows by induction that every M_∞-space is the image of a metric space under a proclusion. It now follows from Proposition 2.5 and Theorem 3.4 that every M-space is the image of a metric space under a proclusion. Since every metric space is compactly generated [12], the theorem follows from Proposition 5.1.

In conjunction with k-spaces, an important class of maps is the class of compact-covers. A map $f: X \to Y$ is said to be *compact-covering* if for every compact $Y_0 \subset Y$ there exists a compact $X_0 \subset X$ such that $f(X_0) = Y_0$ [10]. The following two propositions provide a link between k-spaces, compact-covers, and proclusions.

PROPOSITION 5.3 [10]. *If* Y *is a* k-*space and* $f: X \to Y$ *is compact-covering, then* f *is a proclusion.*

PROPOSITION 5.4 [10]. *If* (X, A, f, Y, p) *is a presentation for* Z, *and if* X *and* Y *are paracompact, then* p *is compact-covering.*

We close this section with a lemma which we shall use repeatedly in the sequel.

LEMMA 5.5. *Let* $\{X_n \mid n \in J^+\}$ *be an increasing closed cover of a* k-*space* X. *Then* $X = \sum X_n$ *if and only if each compact subset of* X *lies in some* X_n.

Proof. If each compact subset of X lies in some X_n, then the inclusion-induced map $g: +_n X_n \to X$ of Proposition 2.5 is compact-covering. By Proposition 5.3, g is a proclusion; hence, $X = \sum X_n$, by Proposition 2.5.

The converse follows from Proposition 2.7.

6. SUBSPACES

Suppose that A is a subset of an M-space X. We consider two topologies on A. The first is the classical subset topology—the *inherited topology*—for A. This is defined as the smallest topology under which the inclusion $i: A \to X$ is continuous.

The disadvantage of this topology is that it may not be compactly generated (see Example 6.4), and by Theorem 5.2 it cannot be an M-space.

We therefore consider a second topology—the *subspace topology*—on A. We obtain it by applying the functor k to the inherited topology on A. Consequently, A with the subspace topology is a k-space. Our next principal result (Theorem 6.2) says even more, namely, that A is an M-space.

By a *subspace* of an M-space we mean a subset with the subspace topology. *Unless it is stated otherwise, subsets will have this topology.*

Because restrictions of maps in the topological category are maps and because k is a functor, we have the following result.

PROPOSITION 6.1. *If* A *and* B *are subspaces of* M*-spaces* X *and* Y*, respectively, and if* $f: X \to Y$ *is a map such that* $f(A) \subset B$*, then* $f \mid A: A \to B$ *is continuous.*

THEOREM 6.2. *If* B *is a subspace of an* M*-space (or* M_n*-space)* Z*, then* B *is an* M*-space (or* M_n*-space).*

Proof. If $Z \in M_0$, then $B \in M_0$. Suppose now that every subspace of an M_n-space is an M_n-space and that $Z \in M_{n+1}$. Let (X, A, f, Y, ψ) be an $(n+1)$-presentation for Z. Let

$$X_0 = X \cap \psi^{-1}(B), \qquad Y_0 = Y \cap \psi^{-1}(B), \qquad A_0 = X_0 \cap A, \qquad g = f \mid A_0: A_0 \to Y_0 .$$

The restriction of ψ to $X_0 + Y_0$ defines a map $q: X_0 + Y_0 \to B$ (Proposition 6.1). Because ψ is compact-covering (Proposition 5.4), q is obviously compact-covering, and since B is by definition a k-space, it follows from Proposition 5.3 that q is a proclusion. If $p: X_0 + Y_0 \to X_0 \cup_g Y_0$ is the natural projection, it follows from Corollary 2.2 that $X_0 \cup_g Y_0 \simeq B$. But X_0 is metrizable, and, by the induction hypothesis, $Y_0 \in M_n$. Consequently, $B \in M_{n+1}$. This completes the induction.

Suppose now that $Z \in M$. Let $Z_0, Z_1, \cdots$ be M_∞-spaces such that $Z = \sum Z_n$. By the paragraph above, $Z_n \cap B \in M_\infty$ for each n, and an easy application of Lemma 5.5 shows that $B = \sum (Z_n \cap B)$. Therefore $B \in M$.

It follows from Theorems 5.2 and 6.2 that a subset of an M-space with the inherited topology is an M-space if and only if it is a k-space. If B is a closed subset of a k-space, or an open subset of a regular k-space, then B with the inherited topology is a k-space [12].

COROLLARY 6.3. *If* B *is either a closed or an open subset of an* M*-space (or* M_n*-space)* Z*, then* B *with the inherited topology is an* M*-space (or* M_n*-space), and it coincides with the subspace* B.

We give an example of a subset B of an M-space such that B with the inherited topology is not an M-space.

Example 6.4. S. P. Franklin [5] showed that a space Z is the image of a metric space under a proclusion if and only if Z is *sequential,* that is, if $U \subset Z$ is open provided every sequence converging to a point in U is eventually in U. Consequently, if z is a limit point of Z, then there exists a sequence in $Z - z$ converging to z. In the course of proving Theorem 5.2, we observed that every M-space is the image of a metric space under a proclusion; therefore, every M-space is sequential. Let

$$X = \{(x, y) \in R^2 \mid x > 0,\ y \geq 0\}, \qquad A = \{(x, y) \in X \mid y = 0\}, \qquad Y = \{x \in R^1 \mid x \geq 0\}.$$

The assignment $(x, 0) \to x$ defines a map $f: A \to Y$. Let $p: X + Y \to X \cup_f Y$ be the natural projection, and let $B = X \cup_f Y - p(A)$. It is easily verified that $p(0)$ is a limit point of B, under the topology inherited from $X \cup_f Y$, but that $p(0)$ is not the limit of any sequence in $B - p(0)$. It follows that B, under the inherited topology, is not an M-space. In this example, the functor k on B isolates the point $p(0)$, and therefore the subspace B is the topological sum of the metrizable sets $p(X - A)$ and $p(0)$.

7. PRODUCT SPACES

Because the cartesian product of two k-spaces need not be a k-space, Steenrod has defined the product of two spaces in CG by applying the functor k to the cartesian product [12]. We adopt this definition of the product for M.

PROPOSITION 7.1 [12]. *If* $p: X \to X_0$ *and* $q: Y \to Y_0$ *are proclusions, then* $p \times q: X \times Y \to X_0 \times Y_0$ *is a proclusion.*

THEOREM 7.2. *The product of two* M*-spaces is an* M*-space.*

Proof. We prove first that the product of an M_n-space Z_1 and an M_m-space Z_2 is an M_∞-space. This is trivial if $n + m = 0$, that is, if the factors are metrizable. Suppose we have proved that $Z_1 \times Z_2 \in M_\infty$ whenever $n + m \leq k$, and let $n + m = k + 1$. Then either $n > 0$ or $m > 0$ – say $m > 0$. Let (X, A, f, Y, p) be an m-presentation for Z_2. By the induction hypothesis, $Z_1 \times X$ and $Z_1 \times Y$ are M_∞-spaces. Let $g = 1 \times f: Z_1 \times A \to Z_1 \times Y$. By Lemma 4.2, $(Z_1 \times X) \cup_g (Z_1 \times Y)$ is an M_∞-space, and it follows from Proposition 7.1 and Corollary 2.2 that $(Z_1 \times X) \cup_g (Z_1 \times Y) \cong Z_1 \times Z_2$; therefore $Z_1 \times Z_2 \in M_\infty$.

Now let X and Y be arbitrary M-spaces, and let $X_0, Y_0, X_1, Y_1, \cdots$ be M_∞-spaces such that $X = \sum X_n$ and $Y = \sum Y_n$. By the paragraph above, $X_n \times Y_n \in M_\infty$ for each n, and by an easy application of Lemma 5.5 we see that $X \times Y = \sum (X_n \times Y_n)$. Therefore $X \times Y \in M$.

The product defined above satisfies the axioms for a product in the category CG [12]. It also satisfies the axioms for a product in M; we can easily verify this directly, or by observing that M is a full subcategory of CG.

Although we can extend the definition of the product to any number of factors by applying k to the cartesian product [12], Theorem 7.2 does not extend to infinitely many factors. In fact, we can show that the product of infinitely many nonempty spaces is an M-space if and only if (1) each factor is an M-space, (2) all but countably many of the factors have exactly one point, and (3) all but finitely many of the factors are metrizable.

8. FUNCTION SPACES

Given k-spaces X and Y, denote the set of all maps from X into Y by Y^X. Topologize Y^X by applying the functor k of Section 5 to the compact-open topology. It is not true that $Y^X \in M$ whenever $X \in M$ and $Y \in M$; for example, if X is an uncountable discrete space, then I^X is a Tychonoff cube, which is not an M-space, by Corollary 3.6. The main result of this section is that Y^X is in M whenever X is a compact metric space and Y is a CW-complex. It is not known whether Y^X is in M whenever X is a compact metric space and Y is in M.

We review some of the vocabulary associated with CW-complexes (see also Definition 2.8). Let τ be a triangulation for a CW-complex K, and let K^n $(n \geq 0)$ be the skeleta of K. The elements of τ are called the *cells* of K. An element of K^0 is called a *vertex* of K. If C and D are cells and $\overline{C} \cap D \neq \emptyset$, then D is said to be a *face* of C. If $\tau' \subset \tau$ has the property that every face of every element of τ' is in τ', then the union L of the elements of τ' is called a *subcomplex* of K. The set L is closed in K, and L is a CW-complex in its own right with triangulation τ'. Unions and intersections of subcomplexes are subcomplexes; therefore, for each subset $B \subset K$, there exist a smallest subcomplex B_1 of K containing B and a largest subcomplex B_2 of K disjoint from B. We call B_1 the *complex closure* of B in K, and we call $K - B_2$ the *open star* of B in K (abbreviation: st(B)). The complex closure and open star are closed and open subsets of K, respectively, and both contain B. A cell is *maximal* if it is not a face of any cell other than itself. It follows that a cell C is maximal if and only if K - C is a subcomplex of K (with triangulation $\tau - \{C\}$). If τ is finite, then K is called a *finite* CW-complex. A CW-complex is compact if and only if it is finite. Every finite CW-complex is the complex closure of the union of its maximal cells, and every compact subset of a CW-complex lies in a finite subcomplex.

LEMMA 8.1. (a) *If* X *is a compact metric space and* Y *is metrizable, then* Y^X *is metrizable.*

(b) *If* X *and* Y *are* M-*spaces and* Z *is a closed subspace of* Y, *then the subspace* $\{f \in Y^X \mid f(X) \subset Z\}$ *is closed in* Y^X *and homeomorphic to* Z^X.

(c) *Every finite* CW-*complex is metrizable.*

Proof. (a) Combine [4, Theorem XII, 8.2(3)] with the facts that $M_0 \subset CG$ and k is the identity on CG.

(b) This follows from [4, p. 258, 1.2(b) and Ex. 3] and the fact that every closed set in the compact-open topology is closed in our topology.

(c) This follows from Corollary 3.6 and the fact that every finite CW-complex is compact.

THEOREM 8.2. *If* K *is a* CW-*complex and* X *is a compact metric space, then* K^X *is an* M-*space.*

Proof. Let $\{K_\alpha \mid \alpha \in A\}$ be the collection of all finite subcomplexes of K. For each α, let

$$Y_\alpha = \{f \in K^X \mid f(X) \subset K_\alpha\}.$$

For each positive integer n, let

$$T_n = \{Y_\alpha \mid \text{the number of cells in } K_\alpha \text{ is } n\},$$

and let

$$T^n = \bigcup \{Y_\alpha \mid Y_\alpha \in T_i \text{ for some } i \leq n\};$$

that is, let T^n be the set of all maps whose range lies in a subcomplex having at most n cells. We shall show that each T^n is an M_{n-1}-space and that $K^X = \sum T^n$ $(n \geq 1)$. It will follow that $K^X \in M$.

First observe that $\{T^1, T^2, \cdots\}$ is a cover of K^X. If $f \in K^X$, then, since X is compact, f(X) lies in a finite subcomplex of K. Consequently, $f \in T^n$ for some n.

We now show that T^n is closed in K^X for all n. Let n be fixed, and let $\{f_\beta \mid \beta \in B\}$ be a net in T^n converging to $f \in K^X$. We must show that $f \in T^n$. Let m be the smallest integer such that $f \in T^m$. We show by contradiction that $m \not> n$. Suppose that $m > n$. Let K_α be the complex closure of $f(X)$. The minimality of m implies that K_α has exactly m cells. Let $C_1, \cdots, C_p$ be the maximal cells of K_α; then $f(X) \cap C_i \neq \emptyset$ for all i. Choose a point x_i such that $f(x_i) \in C_i$ $(i = 1, \cdots, p)$. For each i, the set of maps that take x_i into the open star of C_i in K is open in K^X; therefore, for each i there exists an index β_i such that $f_\beta(x_i) \in \operatorname{st}(C_i)$ whenever $\beta \geq \beta_i$. Let $\beta_0 \in B$ be an index greater than each β_i $(1 \leq i \leq p)$. Then $f_{\beta_0}(X)$ meets each $\operatorname{st}(C_i)$. Since K_α is the complex closure of $\bigcup_{i=1}^p C_i$, it follows that the complex closure of $f_{\beta_0}(X)$ contains K_α; and since $m > n$, we have the contradiction $f_{\beta_0} \notin T^n$. We conclude that $m \leq n$. Thus, we have the relations $f \in T^m \subset T^n$; hence T^n is closed.

We show next that $T^n \in M_{n-1}$ for all n. If $Y_\alpha \in T_1$, then K_α is a vertex of K. Therefore T^1 is discrete. Assume that $T^n \in M_{n-1}$, and consider T^{n+1}. For each $Y_\alpha \in T_{n+1}$, the set $Z_\alpha = Y_\alpha \cap T^n$ is closed in Y_α, by the paragraph above. Let $f_\alpha \colon Z_\alpha \to T^n$ be the inclusion. If Y is the topological sum of all the Y_α in T_{n+1}, and if Z is the topological sum of the corresponding sets Z_α, then (Y, Z) is a pair, and the functions $\{f_\alpha\}$ define a map $f\colon Z \to T^n$. Since each $Y_\alpha \in T_{n+1}$ is metrizable (by parts (a) and (c) of Lemma 8.1), Y is also metrizable; therefore, by the induction hypothesis, $Y \cup_f T^n$ is an M_n-space.

We shall show that $Y \cup_f T^n$ is homeomorphic to T^{n+1}. Since $Y_\alpha \subset T^{n+1}$ for all $Y_\alpha \in T_{n+1}$, and since $T^n \subset T^{n+1}$, there exists an inclusion-induced map $q\colon Y + T^n \to T^{n+1}$. It is compact-covering. To see this, suppose $E \subset T^{n+1}$ is compact. If $E \subset T^n$, then, since $q \mid T^n$ is an inclusion, E is the image of itself under q. Suppose then that $E \not\subset T^n$. Since the evaluation $e\colon X \times K^X \to K$ is continuous [12], $e(X \times E)$ is compact, and therefore it lies in a finite subcomplex L of K. Let $\{K_{\alpha_1}, \cdots, K_{\alpha_j}\}$ be the collection of all subcomplexes of L with exactly $n+1$ cells. (This collection is not empty, because $E \not\subset T^n$.) Then $E \subset \bigcup_{i=1}^j Y_{\alpha_i}$. Since K_{α_i} is closed in K, Y_{α_i} is closed in K^X (by part (b) of Lemma 8.1); therefore $E \cap Y_{\alpha_i}$ is compact. Moreover, $Y_{\alpha_i} \in T_{n+1}$ for all i, and therefore $q \mid Y_{\alpha_i}$ is a homeomorphism (into). Consequently, $(q \mid Y_{\alpha_i})^{-1}(E \cap Y_{\alpha_i})$ is compact for all i. Writing

$$q\left(\bigcup_{i=1}^{j} (q \mid Y_{\alpha_i})^{-1}(E \cap Y_{\alpha_i})\right) = \bigcup_{i=1}^{j} (E \cap Y_{\alpha_i}) = E \cap \left(\bigcup_{i=1}^{j} Y_{\alpha_i}\right) = E,$$

we see that q is compact-covering. By Proposition 5.3, q is a proclusion. Since q and the natural projection $p\colon Y + T^n \to Y \cup_f T^n$ satisfy the hypothesis of Corollary 2.2, it follows that $T^{n+1} \cong Y \cup_f T^n$. Therefore T^{n+1} is an M_n-space, and the induction is complete.

Finally, we show that $K^X = \sum T^n$. We have already observed that $\{T^n \mid n \geq 1\}$ is an increasing closed cover of K^X, and, by an argument similar to the one above involving the evaluation map, we see that every compact subset of K^X lies in some T^n. By Lemma 5.5, $K^X = \sum T^n$, and the proof is complete.

COROLLARY 8.3. *If* K *is a* CW-*complex and* X *and* Y *are compact metric spaces, then* $(K^X)^Y$ *and* $K^{X \times Y}$ *are homeomorphic* M-*spaces.*

Proof. $(K^X)^Y$ and $K^{X \times Y}$ are homeomorphic [12]. By Theorem 8.2, $K^{X \times Y}$ is an M-space; therefore $(K^X)^Y$ is an M-space.

9. SEPARABLE M-SPACES

Among the pathological examples of topology, there are separable spaces that are not Lindelöf spaces, nonseparable Lindelöf spaces, nonseparable subsets of separable spaces, and non-Lindelöf subsets of Lindelöf spaces. The results of this section show that such examples cannot occur in M.

A regular Hausdorff space Y is called an $\aleph_0$-*space* if there exist a separable metric space X and a compact-cover f: $X \to Y$. (This definition differs from that given in [10]; but it is shown in [10] that the two definitions are equivalent.)

THEOREM 9.1. *The following statements concerning an* M-*space* Z *are equivalent:*

(a) Z *is separable;*

(b) Z *is a Lindelöf space;*

(c) Z *is an* $\aleph_0$-*space.*

Proof. (a) $\to$ (b). Combine Theorem 3.5 with the fact that every separable paracompact space is a Lindelöf space [4, p. 176].

(c) $\to$ (a). This follows from the fact that separability is a continuous invariant.

(b) $\to$ (c). Let Z be a Lindelöf M_∞-space. If $Z \in M_0$, then Z is a separable metric space, and therefore an $\aleph_0$-space. Suppose we have shown that all Lindelöf M_n-spaces are $\aleph_0$-spaces, and suppose that $Z \in M_{n+1}$. Let (X, A, f, Y, p) be an (n + 1)-presentation for Z. Since p(Y) is closed in Z (Proposition 2.3), p(Y) is a Lindelöf space; by the induction hypothesis, Y is an $\aleph_0$-space. Since p(X - A) is open in Z (Proposition 2.3), p(X - A) is an F_σ-set in Z, by Theorem 3.5; therefore p(X - A) is a Lindelöf space. It follows that $T = \overline{(X - A)} \subset X$ is a separable metric space and hence an $\aleph_0$-space. Let $g = f \mid T \cap A$. By [10, Theorem H], $T \cup_g Y$ is an $\aleph_0$-space, and, by Proposition 2.4, $T \cup_g Y \cong X \cup_f Y$. Therefore Z is an $\aleph_0$-space, and, by induction, all Lindelöf M_∞-spaces are $\aleph_0$-spaces.

Now suppose that Z is an arbitrary Lindelöf M-space. Let $Z_0, Z_1, \cdots$ be M_∞-spaces such that $Z = \sum Z_n$. Since Z_n is closed in Z, Z_n is a Lindelöf space. By the paragraph above, Z_n is an $\aleph_0$-space for each n. It now follows from Lemma 5.5 and [10, Proposition 7.7] that Z is an $\aleph_0$-space.

COROLLARY 9.2. *If* Z *is a separable* M-*space, then every subspace of* Z *is separable.*

Proof. By [10, Theorems E and I], a subspace A of Z is an $\aleph_0$-space if A is regular. But A is regular, by Theorem 3.5.

Remark. Every $\aleph_0$-space satisfying the first countability axiom is metrizable [10]; consequently, every separable M-space satisfying the first countability axiom is metrizable. This leads to the question: Is every M-space that satisfies the first countability axiom metrizable? Borges [2] has announced an affirmative answer.

10. EXTENSIONS OF MAPPINGS

Many of the interesting results in the theory of retracts and extensions of mappings for metric spaces carry over to M-spaces. In this section, we shall establish some of these results.

We recall some definitions from the theory of retracts. Let (X, A) be a pair. If there exists a map $r: X \to A$ such that $r(a) = a$ for all $a \in A$, then A is said to be a *retract* of X, and r is called a *retraction* of X onto A. If A is a retract of some neighborhood U of itself in X, then A is said to be a *neighborhood retract* of X. A retraction $r: U \to A$ is called a *neighborhood retraction* in X.

Let $\mathscr{C}$ be a category of spaces and maps. A space Y (not necessarily in $\mathscr{C}$) is called an *absolute extensor* for $\mathscr{C}$ (abbreviation: AE($\mathscr{C}$)) if for each pair (X, A) with $X \in \mathscr{C}$ each map $f: A \to Y$ has an extension $F: X \to Y$. A $\mathscr{C}$-space Y is called an *absolute retract* for $\mathscr{C}$ (abbreviation: AR($\mathscr{C}$)) if it is a retract of every $\mathscr{C}$-space that contains it as a closed subset. Clearly, if $Y \in \mathscr{C}$ and Y is an AE($\mathscr{C}$), then Y is an AR($\mathscr{C}$). For many classes $\mathscr{C}$, every AR($\mathscr{C}$) is an AE($\mathscr{C}$). We shall see that every AR(M) is an AE(M) (Theorem 10.2).

A space Y is called an *absolute neighborhood extensor* for $\mathscr{C}$ (abbreviation: ANE($\mathscr{C}$)) if for each pair (X, A) with $X \in \mathscr{C}$ each map $f: A \to Y$ has a neighborhood extension $F: U \to Y$. A $\mathscr{C}$-space Y is called an *absolute neighborhood retract* for $\mathscr{C}$ (abbreviation: ANR($\mathscr{C}$)) if it is a neighborhood retract of every $\mathscr{C}$-space that contains it as a closed subset. Remarks analogous to those above for AE's and AR's hold for ANE's and ANR's.

THEOREM 10.1. *A space* S *is an* AE(M) *if and only if it is an* AE(M_0).

The theorem asserts that the classes of absolute extensors are unable to distinguish between M_0 and M. In this sense, M_0 is "dense" in M; in other words, M is only "slightly larger" than M_0.

Proof. Since M contains all metric spaces, the necessity is trivial. Conversely, assume S is an AE(M_0), and suppose we have proved that S is an AE(M_n). Let (Z, B) be a pair, with $Z \in M_{n+1}$, and let $g: B \to S$ be a map. Let (X, A, f, Y, p) be an (n + 1)-presentation for Z. Identifying Y with p(Y) (Proposition 2.3), we can (by the induction hypothesis) extend $g \mid B \cap Y$ to a map $h: Y \to S$. Let $D = X \cap p^{-1}(B)$, and let $\pi: D \to B$ be the restriction of p. The maps $hf: A \to S$ and $g\pi: D \to S$ together define a map $(hf \cup g\pi): A \cup D \to S$, which extends to a map $\psi: X \to S$. Together, ψ and h induce a map $G: X \cup_f Y \to S$ [4, p. 129], which extends g. By induction, S is an AE(M_∞).

Assume now that (Z, B) is a pair, with $Z \in M$, and let $g: B \to S$ be a map. There exist M_∞-spaces $Z_0, Z_1, \cdots$ such that $Z = \sum Z_n$. For each n, let $B_n = B \cap Z_n$, and let $g_n = g \mid B_n$. Since S is an AE(M_∞), g_0 admits an extension $G_0: Z_0 \to S$. Assume that maps $G_n: Z_n \to S$ $(0 \le n \le k)$ have been defined such that G_{n+1} extends both G_n and g_{n+1} $(n < k)$. Extend the map

$$G_k \cup g_{k+1}: Z_k \cup B_{k+1} \to S$$

to a map $G_{k+1}: Z_{k+1} \to S$. By induction, we obtain a sequence of maps $G_n: Z_n \to S$ $(n \in J^+)$ such that G_{n+1} extends both G_n and g_{n+1} for all n. By Proposition 2.6, $G = \bigcup_n G_n: Z \to S$ is continuous. Since G extends g, the proof is complete.

THEOREM 10.2. *An* M-*space is an* ANR(M) (*or* AR(M)) *if and only if it is an* ANE(M) (*or* AE(M)).

Proof. Since M is closed under adjunction, the theorem follows as in [7, Theorem III, 3.2 Case I].

COROLLARY 10.3. *A neighborhood retract of an* ANR(M) *is an* ANR(M). *A retract of an* AR(M) *is an* AR(M).

Proof. If $\mathscr{C}$ is any category, then a neighborhood retract of an ANE($\mathscr{C}$) is an ANE($\mathscr{C}$), and a retract of an AE($\mathscr{C}$) is an AE($\mathscr{C}$) [7, pp. 40-41].

Because we shall work almost exclusively with M-spaces, we shall simply write AR in place of AR(M). The general usage of "AR" is the abbreviation of "absolute retract for metric spaces" (AR(M_0)). Fortunately, our usage is consistent with this for (by Theorems 10.1 and 10.2) a space is an AR(M_0) if and only if it is a metrizable AR. Therefore we are not modifying but extending the classical notion from the metric category to the category M. We also write ANR instead of ANR(M), and we can easily show that a space is an ANR(M_0) if and only if it is a metrizable ANR.

THEOREM 10.4. *If* $Y = \sum Y_n$ *and* Y_n *is an* AR *for each* n, *then* Y *is an* AR.

Proof. By Theorems 10.1 and 10.2, it is sufficient to show that Y is an AE(M_0). Suppose then that (X, A) is a pair, with $X \in M_0$, and that $f: A \to Y$ is a map. Let d be a metric for X. Without loss of generality, we may assume that $f(A) \cap Y_0 \neq \emptyset$. For each n, let $A_n = f^{-1}(Y_n)$; then $A_n \neq \emptyset$. Let

$$X_n = \begin{cases} \{x \in X \mid d(x, A_n) \leq n \cdot d(x, A - A_n)\} & \text{if } A \neq A_n, \\ X & \text{if } A = A_n. \end{cases}$$

Arguing as in the proof of Lemma 4.2, we see that $X = \sum X_n$.

Since Y_0 is an AR and $f(A_0) \subset Y_0$, there exists an extension $F_0: X_0 \to Y_0$ of $f \mid A_0$. Assume that maps $F_n: X_n \to Y_n$ $(0 \leq n \leq k)$ have been defined such that F_{n+1} extends both F_n and $f \mid A_{n+1}: A_{n+1} \to Y_{n+1}$ $(n < k)$. Since $X_k \cap A = A_k$, and since F_k and f agree on A_k,

$$F_k \cup (f \mid A_{k+1}): X_k \cup A_{k+1} \to Y_{k+1}$$

is a well-defined map; since Y_{k+1} is an AR, this map extends to a map $F_{k+1}: X_{k+1} \to Y_{k+1}$. Repeating this argument, we obtain a sequence of maps $F_n: X_n \to Y_n$ $(n \in J^+)$, each F_{n+1} extending both F_n and $f \mid A_{n+1}$. By Proposition 2.6, $F = \bigcup_n F_n: X \to Y$ is continuous. Since F extends f, the theorem is proved.

Many categories $\mathscr{C}$ have the property that every $\mathscr{C}$-space Z can be embedded as a closed subset in a $\mathscr{C}$-space Z_0, where Z_0 is an AR($\mathscr{C}$). We shall show that M has this property (Theorem 10.8).

LEMMA 10.5. *Let* (X, A) *be a pair, where* X *is metrizable, and let* $f: A \to Y$ *be a map. If* X, A, *and* Y *are* AR's, *then* $X \cup_f Y$ *is an* AR.

Proof. By [8, Lemmas 3.1 and 3.3 and Theorem 4.2], $X \cup_f Y$ is an AE(M_0). The result now follows from Theorems 10.1 and 10.2.

LEMMA 10.6. *Every* M_n-*space* Z *can be embedded as a closed subset in an* M_n-*space* Z_0, *where* Z_0 *is an* AR.

Proof. This result is known for metrizable spaces [7, Theorems II, 14.1 and III, 2.1]. Assume that we have proved it for all M_n-spaces, and suppose Z is an M_{n+1}-space. Let (X, A, f, Y, p) be an $(n+1)$-presentation for Z. Embed A as a closed subset of a metric space A_0, where A_0 is an AR. We can choose A_0 so that $A_0 \cap X = A$; then $A_0 \cup X$ with the union topology (see Section 4) is metrizable. Embed $A_0 \cup X$ as a closed subset in a metric space X_0, where X_0 is an AR. By the induction hypothesis, we may embed Y as a closed subset in an M_n-space Y_0, where Y_0 is an AR. Extend $f: A \to Y$ to a map $g: A_0 \to Y_0$. By Lemma 10.5, $X_0 \cup_g Y_0$ is an AR, and, by Proposition 2.4, $X \cup_f Y$ is homeomorphic to a closed subset of $X_0 \cup_g Y_0$. This completes the induction.

LEMMA 10.7. *If* $Y_0, Y_1, \cdots$ *are subsets of an* M-*space* Y *such that* $Y = \sum Y_n$, *then there exist an* M-*space* Z *containing* Y *as a closed subset, and* M_∞-*spaces* $Z_0, Z_1, \cdots$ *such that*

(1) $Z = \sum Z_n$,

(2) Z_n *is an* AR *for all* n,

(3) Y_n *is a closed subset of* Z_n *for all* n, *and*

(4) $Z_n \cap Y = Y_n$ *for all* n.

Proof. By Lemma 10.6, we can embed Y_0 as a closed subset in an M_∞-space Z_0, where Z_0 is an AR. We can choose Z_0 so that $Z_0 \cap Y = Y_0$. Assume that M_∞-spaces $Z_0, \cdots, Z_k$ have been defined such that Z_n is closed in Z_{n+1} for all $n < k$ and such that (2) to (4) hold for all $n \leq k$. By the remark following Lemma 4.2, $Z_k \cup Y_{k+1}$ with the union topology is an M_∞-space. Therefore, by Lemma 10.6, we can embed $Z_k \cup Y_{k+1}$ as a closed subset in an M_∞-space Z_{k+1}, where Z_{k+1} is an AR, and we can choose Z_{k+1} so that $Z_{k+1} \cap Y = Y_{k+1}$. By induction, we obtain an infinite sequence of M_∞-spaces $Z_0, Z_1, \cdots$ satisfying (2) to (4). The result now follows if we take $Z = \sum Z_n$ (see Remark 2 of Section 2).

THEOREM 10.8. *Every* M-*space* Z *can be embedded as a closed subset in an* M-*space* Z_0, *where* Z_0 *is an* AR.

Proof. This follows at once from Lemma 10.7 and Theorem 10.4.

We shall close this section with a result (Theorem 10.10) concerning products of ANR's and AR's.

LEMMA 10.9. *A Hausdorff space* Y *is an* ANE(M) (*or* AE(M)) *if and only if* kY *is an* ANE(M) (*or* AE(M)), *where* k *is the functor defined in Section* 5.

Proof. Let $g: kY \to Y$ be the identity, and let (X, A) be a pair, with $X \in M$.

Assume first that Y is an ANE(M), and let $f: A \to kY$ be a map. The map $gf: A \to Y$ has a neighborhood extension $F: U \to Y$ in X, and $kF: kU = U \to kY$ is a neighborhood extension of f; therefore kY is an ANE(M).

Conversely, assume that kY is an ANE(M), and let $f: A \to Y$ be a map. The map $kf: kA = A \to kY$ has a neighborhood extension $F: U \to kY$ in X, and $gF: U \to Y$ is a neighborhood extension of f; therefore Y is an ANE(M).

A similar argument holds for AR's.

THEOREM 10.10. *The product* (*as defined in Section* 7) *of two* ANR's (*or* AR's) *is an* ANR (*or* AR).

Proof. The cartesian product of two ANR's (or AR's) is an ANE(M) (or AE(M)) [7, pp. 39-40]. The result now follows from Lemma 10.9 and Theorem 10.2.

11. ADJUNCTIONS OF ANR's

Let (X, A) be a pair, and let $f: A \to Y$ be a map, where X, A, and Y are metrizable ANR's. If $X \cup_f Y$ is metrizable, then it is an ANR [7, p. 178]. This result was obtained in successive stages by Borsuk [3], Whitehead [13], and Hanner [6]. In this section, we shall extend the result to the category M.

THEOREM 11.1. *Suppose that* (X, A) *is a pair and* $f: A \to Y$ *is a map. If* X, A, *and* Y *are* ANR's (*or* AR's), *then* $X \cup_f Y$ *is an* ANR (*or* AR).

We shall prove this theorem in several steps. First we shall prove the statement for AR's (Step 4), and then we shall apply it to prove the statement for ANR's (Step 6).

Let (X, A) be a pair. We say that (X, A) is a *proper* 0-*pair* if both X and A are metrizable AR's; we say that (X, A) is a *proper* n-*pair* ($n > 0$) if there exists an n-presentation (Z, B, g, E, p) for X such that each of

$$E, \quad E \cap p^{-1}(A), \quad Z, \quad Z \cap p^{-1}(A), \quad B, \quad B \cap p^{-1}(A)$$

is an AR. We call (Z, B, g, E, p) a *proper* n-*presentation* for X with respect to A.

STEP 1. *If* (X, A) *is a proper* n-*pair* ($n \geq 0$), *then* X *and* A *are* AR's.

Proof. If $n = 0$, then X and A are AR's, by the definition of a proper 0-pair. If $n > 0$ and (Z, B, g, E, p) is a proper n-presentation for X with respect to A, then Z, B, and E are AR's; therefore, by Lemma 10.5, $X \simeq Z \cup_g E$ is an AR. To show that A is an AR, let

$$Z' = Z \cap p^{-1}(A), \qquad B' = B \cap p^{-1}(A), \qquad E' = E \cap p^{-1}(A),$$

and let $h: B' \to E'$ be the restriction of g. Then Z', B', and E' are AR's, and, by Proposition 2.4, $A \simeq Z' \cup_h E'$. Therefore, A is an AR, by Lemma 10.5.

STEP 2. *Every pair* (X, A) *with* $X \in M_n$ *can be embedded as a closed subpair in a proper* n-*pair* (X_0, A_0).

Proof. Suppose that $X \in M_0$. Embed A as a closed subset in a space A_0, where A_0 is a metrizable AR. We can choose A_0 so that $A = A_0 \cap X$. Since $A_0 \cup X$ is metrizable under the union topology, we can embed it as a closed subset in X_0, a metrizable AR. Thus (X, A) is a closed subpair of the proper 0-pair (X_0, A_0).

Assume that we have proved the statement for M_n-spaces, and suppose $X \in M_{n+1}$. Let (Z_1, B_1, h, E_1, q) be an $(n+1)$-presentation for X. By the induction hypothesis, we can embed $(E_1, E_1 \cap q^{-1}(A))$ as a closed subpair in a proper n-pair (E_0, D_0). By Step 1, E_0 and D_0 are AR's. Embed $B_1 \cap q^{-1}(A)$ as a closed subset in B_2, where B_2 is a metrizable AR such that

$$(1) \qquad Z_1 \cap B_2 = B_1 \cap q^{-1}(A).$$

Under the union topology, $Z_1 \cup B_2$ is metrizable; embed $B_1 \cup B_2$ as a closed subset in B, a metrizable AR such that

(2) $$B \cap (Z_1 \cup B_2) = B_1 \cup B_2\,;$$

embed $B_2 \cup (Z_1 \cap q^{-1}(A))$ as a closed subset in B_3, a metrizable AR such that

(3) $$B_3 \cap (Z_1 \cup B) = B_2 \cup (Z_1 \cap q^{-1}(A))\,;$$

finally, embed $Z_1 \cup B \cup B_3$ (with the union topology) as a closed subset in Z, a metrizable AR. By (1), (2), and (3) we see that

(4) $$B_2 = B \cap B_3\,.$$

Since $h(B_1 \cap q^{-1}(A)) \subset D_0$, and since E_0 and D_0 are AR's, h admits an extension $H: B \to E_0$ such that $H(B_2) \subset D_0$. Since B is metrizable, there exists a map $\lambda: B \to I$ such that

(5) $$\lambda^{-1}(0) = B_1 \cup B_2\,.$$

Let $E = E_0 \times I$, and identify E_0 with $E_0 \times \{0\} \subset E$. Define a map $g: B \to E$ by

$$g(b) = (H(b), \lambda(b)) \qquad \text{for all } b \in B\,.$$

Let $X_0 = Z \cup_g E$, and let $A_0 = p(B_3 + D_0)$, where $p: Z + E \to X_0$ is the natural projection. By Proposition 2.4, A_0 is closed in X_0. We shall show that

(*) (Z, B, g, E, p) is a proper $(n+1)$-presentation for X_0 with respect to A_0.

It follows from (5) that $B_2 = g^{-1}(D_0)$; combining this with (4), we see that

$$E \cap p^{-1}(A_0) = D_0\,,$$

$$Z \cap p^{-1}(A_0) = B_3\,,$$

$$B \cap p^{-1}(A_0) = B_2\,.$$

D_0, B_3, B_2, Z, B, and E_0 are AR's, by their definitions, and by Theorem 10.10, $E = E_0 \times I$ is an AR; the assertion (*) follows.

We shall now show that (X, A) is homeomorphic to a closed subpair of (X_0, A_0). Since $(E_1, E_1 \cap q^{-1}(A))$ is a subpair of (E_0, D_0), we see that

$$E_1 \cap D_0 = E_1 \cap q^{-1}(A)\,.$$

By (1) and (3),

$$Z_1 \cap B_3 = Z_1 \cap q^{-1}(A)\,.$$

Since $B_2 = g^{-1}(D_0)$, it follows from (4) that $B_3 + D_0$ is saturated with respect to p; therefore, we may write

(6) $$\begin{aligned} p(Z_1 + E_1) \cap p(B_3 + D_0) &= p(Z_1 \cap B_3 + E_1 \cap D_0) \\ &= p(Z_1 \cap q^{-1}(A) + E_1 \cap q^{-1}(A)) = p(q^{-1}(A))\,. \end{aligned}$$

By Proposition 2.4, $p(Z_1 + E_1)$ is closed in X_0, and (X, A) is homeomorphic to the pair $(p(Z_1 + E_1), p(q^{-1}(A)))$. Since $A_0 = p(B_3 + D_0)$, we conclude, by (6), that (X, A) is homeomorphic to a closed subpair of (X_0, A_0). This completes the proof of Step 2.

In Steps 3 *to* 6, *we assume that* (X, A) *is a pair and* f: A → Y *is a map.*

STEP 3. *If* X *is an* M_∞*-space and* X, A, *and* Y *are* AR's, *then* $X \cup_f Y$ *is an* AR.

Proof. If X is metrizable, then $X \cup_f Y$ is an AR, by Lemma 10.5. Assume the result for M_n-spaces, and suppose that $X \in M_{n+1}$. If (X, A) is a proper (n + 1)-pair, let (Z, B, g, E, p) be a proper (n + 1)-presentation. Define a map $h: E \cap p^{-1}(A) \to Y$ by

$$h(x) = fp(x) \qquad \text{for all } x \in E \cap p^{-1}(A).$$

Letting $q: E + Y \to E \cup_h Y$ be the natural projection, define a map

$$j: B \cup (Z \cap p^{-1}(A)) \to E \cup_h Y$$

by

$$j(x) = \begin{cases} qg(x) & \text{if } x \in B, \\ qfp(x) & \text{if } x \in Z \cap p^{-1}(A). \end{cases}$$

Since (Z, B, g, E, p) is a proper (n + 1)-presentation for X with respect to A, the sets E and $E \cap p^{-1}(A)$ are AR's; hence, by the induction hypothesis, $E \cup_h Y$ is an AR. Z, $Z \cap p^{-1}(A)$, B, and $B \cap p^{-1}(A)$ are AR's, and

$$B \cap (Z \cap p^{-1}(A)) = B \cap p^{-1}(A);$$

therefore, by [7, Proposition II, 10.1], $B \cup (Z \cap p^{-1}(A))$ is an AR; hence, by Lemma 10.5, $Z \cup_j (E \cup_h Y)$ is an AR. As in the proof of Lemma 4.2, we see that $Z \cup_j (E \cup_h Y)$ is homeomorphic to $X \cup_f Y$; therefore $X \cup_f Y$ is an AR.

If (X, A) is not proper, embed it as a closed subpair of a proper (n + 1)-pair (X_0, A_0). Let $r: X_0 \to X$ be a retraction such that $r(A_0) \subset A$, and let $g = fr \mid A_0 : A_0 \to Y$. Letting $p: X_0 + Y \to X_0 \cup_g Y$ be the natural projection, define a retraction $s: X_0 \cup_g Y \to p(X + Y)$ by

$$s(x) = \begin{cases} x & \text{if } x \in p(Y), \\ pr(p \mid X_0)^{-1}(x) & \text{if } x \in p(X_0). \end{cases}$$

By Proposition 2.4, p(X + Y) is homeomorphic to $X \cup_f Y$. Since (X_0, A_0) is a proper (n + 1)-pair, and since Y is an AR, $X_0 \cup_g Y$ is an AR, by the discussion in the preceding paragraph. Therefore $X \cup_f Y$ is an AR, by Corollary 10.3, and the induction is complete.

STEP 4. *If* X, A, *and* Y *are* AR's, *then* $X \cup_f Y$ *is an* AR.

Proof. Let $X_0, X_1, \cdots$ be M_∞-spaces such that $X = \sum X_n$. For each n, let $A_n = A \cap X_n$. By Lemma 10.7, there exist an M-space B and M_∞-spaces $B_0, B_1, \cdots$ such that

(1) $B = \sum B_n$,

(2) B_n is an AR for all n,

(3) A_n is a closed subset of B_n for all n, and

(4) $B_n \cap A = A_n$ for all n.

In addition, we may assume that

(5) $X \cap B = A$.

Putting the union topology on $X \cup B$, we see, by Lemma 5.5, that

$$X \cup B = \sum (X_n \cup B_n);$$

therefore, by Lemma 10.7, there exist an M-space Z and M_∞-spaces $Z_0, Z_1, \cdots$ such that

(6) $Z = \sum Z_n$,

(7) Z_n is an AR for all n,

(8) $X_n \cup B_n$ is a closed subset of Z_n for all n, and

(9) $Z_n \cap (X \cup B) = X_n \cup B_n$ for all n.

By (5), (X, A) is a (closed) subpair of (Z, B). Since X and A are AR's, there exists a retraction $r: Z \to X$ such that $r(B) \subset A$. Let $g = fr \mid B: B \to Y$, and $g_n = g \mid B_n$. By Step 3, $Z_n \cup_{g_n} Y$ is an AR, and, by Proposition 2.4, $Z_n \cup_{g_n} Y$ is homeomorphic to $p(Z_n + Y)$, where $p: Z + Y \to Z \cup_g Y$ is the natural projection. By Lemma 4.1, $Z \cup_g Y = \sum p(Z_n + Y)$; hence, $Z \cup_g Y$ is an AR, by Theorem 10.4. Define a retraction $s: Z \cup_g Y \to p(X + Y)$ by

$$s(x) = \begin{cases} x & \text{if } x \in p(Y), \\ pr(p \mid Z)^{-1}(x) & \text{if } x \in p(Z). \end{cases}$$

By Corollary 10.3, $p(X + Y)$ is an AR, and, by Proposition 2.4, $p(X + Y)$ is homeomorphic to $X \cup_f Y$. Therefore $X \cup_f Y$ is an AR.

STEP 5. *If* X *and* A *are* ANR's *and* Y *is an* AR, *then* $X \cup_f Y$ *is an* ANR.

Proof. By the methods of Step 4, we can embed (X, A) as a closed subpair in a pair (X_0, A_0), where X_0 and A_0 are AR's. Since X and A are ANR's, there exist a neighborhood U of X in X_0 and a retraction $r: \overline{U} \to X$ such that $r(\overline{U} \cap A_0) \subset A$. Since Y is an AR, the map

$$fr \mid \overline{U} \cap A_0 : \overline{U} \cap A_0 \to Y$$

admits an extension $F: A_0 \to Y$. By the perfect normality of A_0, there exists a map $\lambda: A_0 \to I$ such that

$$\lambda^{-1}(0) = A \quad \text{and} \quad \lambda^{-1}(1) = A_0 - U.$$

Define a map $g: A_0 \to Y \times I = Y_0$ by

$$g(a) = (F(a), \lambda(a)) \quad \text{for all } a \in A_0.$$

Since X_0, A_0, and Y_0 are AR's (Y_0 is an AR by Theorem 10.10), we see by Step 4 that $X_0 \cup_g Y_0$ is an AR. Let $p: X_0 + Y_0 \to X_0 \cup_g Y_0$ be the natural projection. The condition $\lambda^{-1}(1) = A_0 - U$ guarantees that the open subset $U + (Y \times [0, 1))$ of $X_0 + Y_0$ is saturated with respect to p; therefore $E = p(U) \cup p(Y \times [0, 1))$ is open in $X_0 \cup_g Y_0$. Identifying Y with $Y \times \{0\} \subset Y_0$, let $\pi: Y_0 \to Y$ be the coordinate projection. The map $s: E \to p(X + Y)$ defined by

$$s(x) = \begin{cases} p\pi(p \mid Y_0)^{-1}(x) & \text{if } x \in p(Y \times [0, 1)), \\ pr(p \mid U)^{-1}(x) & \text{if } x \in p(U) \end{cases}$$

retracts E onto $p(X + Y)$. By Corollary 10.3, $p(X + Y)$ is an ANR. By the formula $\lambda(A) = 0$, and by the identification $Y \equiv Y \times \{0\} \subset Y_0$, we see that g extends f; therefore Proposition 2.4 is applicable and shows that $X \cup_f Y$ is homeomorphic to $p(X + Y)$. Consequently, $X \cup_f Y$ is an ANR.

STEP 6. *If* X, A, *and* Y *are* ANR's, *then* $X \cup_f Y$ *is an* ANR.

Proof. Embed Y as a closed subset in Z, an AR, and let $r: U \to Y$ be a neighborhood retraction in Z. If we consider f to be a map from A into Z and U as well as into Y, then the inclusions $Y \subset U \subset Z$ induce inclusions

$$X \cup_f Y \subset X \cup_f U \subset X \cup_f Z.$$

$X \cup_f U$ is open in $X \cup_f Z$. Letting $p: X + U \to X \cup_f U$ be the natural projection, define a retraction $s: X \cup_f U \to X \cup_f Y$ by

$$s(x) = \begin{cases} x & \text{if } x \in p(X), \\ pr(p \mid U)^{-1}(x) & \text{if } x \in p(U). \end{cases}$$

By Step 5, $X \cup_f Z$ is an ANR. Therefore $X \cup_f Y$ is an ANR, by Corollary 10.3, and the proof of Theorem 11.1 is complete.

Recall that the (unreduced) cone CY over a space Y is the quotient $Y \times I/Y \times \{1\}$. CY is homeomorphic to the adjunction space $(Y \times I) \cup_f Z$, where Z consists of a single point and $f: Y \times \{1\} \to Z$ is the unique map. Identifying CY with $(Y \times I) \cup_f Z$, and letting $p: (Y \times I) + Z \to CY$ be the natural projection, we call the point $p(Y \times \{1\})$ the *vertex* of CY, and we identify Y with the *base* $p(Y \times \{0\})$. The results of Sections 4, 6, and 7 show that $Y \in M$ if and only if $CY \in M$.

THEOREM 11.2. Y *is an* ANR *if and only if* CY *is an* AR.

Proof. If Y is an ANR, then, by Theorem 10.10, $Y \times I$ is an ANR, and, by Theorem 11.1, CY is an ANR. Since CY is contractible, it is an AR [7, p. 43]. Conversely, since Y is a neighborhood retract of CY (delete the vertex and project vertically onto the base), we see by Corollary 10.3 that Y is an ANR if CY is an AR.

THEOREM 11.3. *If* $Y = \sum Y_n$ *and* Y_n *is an* ANR *for each* n, *then* Y *is an* ANR.

Proof. We can obtain this from a result of Kodama [9], but it also follows quite simply from Theorem 11.2. By Theorem 11.2, $C(Y_n)$ is an AR; therefore $\sum C(Y_n)$ is an AR, by Theorem 10.4. But $\sum C(Y_n)$ and CY are homeomorphic; therefore, Y is an ANR, by Theorem 11.2.

An important result in homotopy theory is that every metrizable CW-complex is an ANR. We conclude this paper by generalizing this result.

THEOREM 11.4. *Every* CW-*complex* K *is an* ANR.

Proof. By [7, Theorem II, 17.2], a free union of cells and spheres is an ANR. With the help of Theorem 11.1 and an easy induction, we can show that K^n is an ANR for all n. By Theorem 11.3, $K = \sum K^n$ is an ANR.

REFERENCES

1. C. J. R. Borges, *On stratifiable spaces.* Pacific J. Math. 17 (1966), 1-16.
2. ———, *A note on the* M-*spaces of Hyman.* Abstract 67T-560, Notices Amer. Math. Soc. 14 (1967), 823.
3. K. Borsuk, *Über eine Klasse von lokal zusammenhängenden Räumen.* Fund. Math. 19 (1932), 220-242.
4. J. Dugundji, *Topology.* Allyn and Bacon, Boston, Mass., 1966.
5. S. P. Franklin, *Spaces in which sequences suffice.* Fund. Math. 57 (1965), 107-115.
6. O. Hanner, *Some theorems on absolute neighborhood retracts.* Ark. Mat. 1 (1951), 389-408.
7. S.-T. Hu, *Theory of retracts.* Wayne State Univ. Press, Detroit, Mich., 1965.
8. D. M. Hyman, *A generalization of the Borsuk-Whitehead-Hanner theorem.* Pacific J. Math. 23 (1967), 263-271.
9. Y. Kodama, *Note on an absolute neighborhood extensor for metric spaces.* J. Math. Soc. Japan 8 (1956), 206-215.
10. E. Michael, $\aleph_0$-*spaces.* J. Math. Mech. 15 (1966), 983-1002.
11. E. H. Spanier, *Algebraic topology.* McGraw-Hill, New York, 1966.
12. N. E. Steenrod, *A convenient category of topological spaces.* Michigan Math. J. 14 (1967), 133-152.
13. J. H. C. Whitehead, *Note on a theorem due to Borsuk.* Bull. Amer. Math. Soc. 54 (1948), 1125-1132.

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