AN EXTREMAL PROBLEM FOR FUNCTIONS WITH POSITIVE REAL PART

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1. INTRODUCTION

In a recent paper [5] the author established the following solution of an extremal problem. Let $\mathfrak{P}$ denote the class of regular functions $P(z)$ in the unit disc E $(|z| < 1)$ with $P(0) = 1$ and $\Re P(z) > 0$. Let $F(w_1, w_2)$ be a given function analytic in the half-plane $\Re w_1 > 0$ and in the w_2-plane. Then for every r $(0 < r < 1)$ the value of

$$\min_{P \in \mathfrak{P}} \min_{|z|=r} \Re F(P(z), zP'(z))$$

occurs only for an extremal function of the form

$$P(z) = \sum_{n=1}^{2} \lambda_n \left(\frac{1 + \varepsilon_n z}{1 - \varepsilon_n z}\right),$$

where

$$|\varepsilon_n| = 1, \qquad 0 \leq \lambda_n \leq 1, \qquad \sum_1^2 \lambda_n = 1.$$

The method used variational formulas whose development by the author depends upon the works of Hummel [1], Julia [2], and Schiffer [7]. The same method yields the solution of a related problem in which the term $zP'(z)$ is replaced by

$$\mu(z) = \frac{1}{z}\int_0^z P(t)\,dt \qquad (P(z) \in \mathfrak{P}). \tag{1.1}$$

It should be noticed that $\mu(z) \in \mathfrak{P}$, since

$$\Re \mu(z) = \int_0^1 \Re P(\rho z)\,d\rho > 0 \qquad (z = re^{i\theta},\ r < 1).$$

Specifically, we consider the following problem. Let $F(w)$ denote an arbitrary function regular in the portion D_0 of the half-plane $\Re w > 0$ that is covered by the images of $|z| < 1$ by the mappings $w = \mu(z)$ for $P \in \mathfrak{P}$. As we proceed, D_0 will be explicitly determined. We set

$$m_F(r) = \min_{P \in \mathfrak{P}} \min_{|z|=r} \Re F(\mu(z)) \tag{1.2}$$

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and consider the problem of finding its value by determining the extremal function P(z). The author solved this problem first by the variational method. The method proved to be quite long and somewhat difficult. Its details suggested a much shorter, clearer and almost trivial proof by the method of subordination.

It is the purpose of this paper to present this shorter derivation of the solution of (1.2) (see Theorem 1) and to make applications (Corollaries 1 and 2) to the class $\mathfrak{R}_\alpha$ of analytic functions

$$f(z) = e^{i\alpha} z + a_2 z^2 + \cdots + a_n z^n + \cdots \qquad (\alpha \text{ real and fixed}) \tag{1.3}$$

that are regular and satisfy the condition $\mathfrak{R} f'(z) > 0$ in E. It is well known [4] that such functions f(z) are univalent in E. Recently MacGregor [3] discussed the class $\mathfrak{R}_\alpha$ in the special case $\alpha = 0$. In particular, he pointed out that for $|z| < 1$

$$\frac{1 - |z|}{1 + |z|} \leq \mathfrak{R} f'(z) \leq |f'(z)| \leq \frac{1 + |z|}{1 - |z|}, \tag{1.4}$$

$$- |z| + 2 \log(1 + |z|) \leq |f(z)| \leq - |z| - 2 \log (1 - |z|). \tag{1.5}$$

We obtain the following theorems and corollaries.

THEOREM 1. *Let* $\mathfrak{P}$ *be the class of regular functions* P(z) *in* E ($|z| < 1$) *with* $P(0) = 1$ *and* $\mathfrak{R} P(z) > 0$. *Let* F(w) *denote a nonconstant function that is analytic in the convex domain* D_0 *which is the image of* E *by the mapping*

$$w = \mu_0(z) = \frac{1}{z} \int_0^z \frac{1+t}{1-t}\, dt = -\frac{2}{z}\log(1 - z) - 1.$$

This domain D_0 *lies in the half-strip given by the inequalities* $|\mathfrak{I} w| < \pi$, $\mathfrak{R} w > 2 \log 2 - 1$. *Then, for each* $r > 1$, *the minimum*

$$m_F(r) = \min_{P \in \mathfrak{P}} \min_{|z|=r} \mathfrak{R} F\left(\frac{1}{z}\int_0^z P(t)\, dt\right)$$

occurs for a function of the form $P(z) = (1 + \varepsilon z)(1 - \varepsilon z)^{-1}$, *where* ε *is an arbitrary complex constant of absolute value* 1, *and for no other functions.*

COROLLARY 1. *Let* $\mathfrak{R}_\alpha$ *be the class of regular functions* f(z) *defined on* E *as in* (1.3), *so that* $\mathfrak{R} f'(z) > 0$ *on* E. *Let* G(w) *denote a nonconstant analytic function in the convex domain* D_α *that is the image of* E *by the mapping*

$$w = - e^{-i\alpha} - \frac{2}{z}(\cos \alpha) \log(1 - z).$$

Then, for each fixed $r < 1$, *the minimum*

$$\min_{f \in \mathfrak{R}_\alpha} \min_{|z|=r} \mathfrak{R} G\left(\frac{f(z)}{z}\right)$$

occurs for a function of the form

$$f(z) = -e^{-i\alpha} z - \frac{2}{\varepsilon}(\cos\alpha)\log(1 - \varepsilon z),$$

where ε *is an arbitrary complex constant of absolute value* 1, *and for no other functions.*

The inequalities (1.5), obtained by MacGregor [3] for the class $\mathfrak{R}_0$, are seen to be the special case of Corollary 1 where $G(w)$ is the function $\log w$.

It seems fitting to add the following corollary at this point, since it is also related to the class $\mathfrak{R}_\alpha$. Corollary 2, as will be seen, is an easy consequence of Theorem 2 of the author's earlier paper [5], which states that if $F(w)$ is analytic in the half-plane $\Re w > 0$, then for each $r < 1$

$$\min_{P\in\mathfrak{P}} \min_{|z|=r} \Re F(P(z)) = \min_{|z|=r} \Re F\left(\frac{1+z}{1-z}\right). \tag{1.6}$$

COROLLARY 2. *Let* $\mathfrak{R}_\alpha$ *be the class of regular functions* $f(z)$ *defined on* E *as in* (1.3), *so that* $\Re f'(z) > 0$ *on* E. *Let* $G(w)$ *denote a function that is analytic in the right half-plane* $\Re w > 0$. *Then for each* $r < 1$

$$\min_{f\in\mathfrak{R}_\alpha} \min_{|z|=r} \Re G(f'(z)) = \min_{|z|=r} \Re G\left((\cos\alpha)\frac{1+z}{1-z} + i\sin\alpha\right).$$

The following theorem is closely related to Theorem 1 and will be derived from it.

THEOREM 2. *Let* $\mathfrak{P}$ *be the class described in Theorem* 1. *Let* z_1 *be a complex number* $(0 < |z_1| < 1)$. *Let* $F(w)$ *denote a nonconstant function that is analytic in the convex domain* D_0 *defined as in Theorem* 1. *Then*

$$\min_{P\in\mathfrak{P}} \Re F\left(\frac{1}{z_1}\int_0^{z_1} P(z)\,dz\right) = \Re F\left(\frac{1}{z_1}\int_0^{z_1}\frac{1+\varepsilon z}{1-\varepsilon z}\,dz\right).$$

where ε *depends on* z_1 *and* F, *and* $|\varepsilon| = 1$.

2. A STAR FUNCTION S(z)

Before we proceed to the proof of Theorem 1 it is desirable to discuss the properties of a rather special univalent, starlike function $S(z)$ that will be useful in the proof. The starlike character of $S(z)$ is fundamental to the proof of Theorem 1, whether one uses the method of subordination (which we shall give) or the variational method of proof (which we shall not include). We define $S(z)$ by the equation

$$S(z) = \frac{2}{z}\int_0^z K(t)\,dt, \tag{2.1}$$

where $K(z)$ is the Koebe function $z(1-z)^{-2}$. Thus

$$S(z) = \frac{2}{1-z} + \frac{2}{z}\log(1-z) = \sum_1^\infty \frac{2n}{n+1} z^n. \tag{2.2}$$

$S(z)$ may be written in the useful alternate form

$$S(z) = \frac{1+z}{1-z} - \frac{1}{z}\int_0^z \frac{1+t}{1-t}\,dt = P_0(z) - \mu_0(z), \tag{2.3}$$

where both

$$P_0(z) = \frac{1+z}{1-z} \quad \text{and} \quad \mu_0(z) = \frac{1}{z}\int_0^z P_0(t)\,dt$$

are members of the class $\mathfrak{P}$. A computation of

$$S(e^{i\theta}) = U(\theta) + iV(\theta)$$

for $0 < \theta \leq \pi$ gives

$$U(\theta) = 1 + (\cos\theta)\log\{2(1-\cos\theta)\} - (\pi-\theta)\sin\theta, \tag{2.4}$$

$$V(\theta) = (\sin\theta)(1-\cos\theta)^{-1} - (\pi-\theta)\cos\theta - \sin\theta\,\log\{2(1-\cos\theta)\} = \frac{dU(\theta)}{d\theta}. \tag{2.5}$$

For $0 < \theta < \pi$, we obtain from (2.3) the inequalities

$$U(\theta) = -\Re\left[\frac{1}{z}\int_0^z \frac{1+t}{1-t}\,dt\right]_{z=e^{i\theta}} = -\int_0^1 \frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2}\,d\rho < 0, \tag{2.6}$$

$$V(\theta) = \frac{dU(\theta)}{d\theta} = \int_0^1 \frac{2\rho(1-\rho^2)\sin\theta}{(1-2\rho\cos\theta+\rho^2)^2}\,d\rho > 0. \tag{2.7}$$

The function $w = S(z)$ maps $|z| = 1$ onto an unbounded curve C, symmetric about the real axis. From (2.6) it follows that C lies in the left half-plane $\Re w < 0$. From (2.7) it follows that the imaginary part of $S(e^{i\theta})$ is positive for $0 < \theta < \pi$ and that its real part is a strictly increasing function of θ. Thus $S(z)$ is both typically-real and univalent in $|z| < 1$.

Moreover, $S(z)$ is starlike with respect to the origin in E. In other words,

$$\Re\,\frac{zS'(z)}{S(z)} > 0 \qquad (|z| < 1). \tag{2.8}$$

Since it follows from (2.1) that

$$zS'(z) + S(z) = K(z) = 2z(1-z)^{-2},$$

we have the relations

$$\frac{zS'(z)}{S(z)} + 1 = \frac{2z}{(1-z)^2 S(z)}, \qquad \frac{zS'(z)}{S(z)} - 1 = \frac{2z - 2(1-z)^2 S(z)}{(1-z)^2 S(z)}.$$

But (2.8) is equivalent to

$$\left|\frac{zS'(z)}{S(z)} - 1\right| < \left|\frac{zS'(z)}{S(z)} + 1\right| \qquad (|z| < 1),$$

in other words, to

$$(2.9)\qquad \left|3z - 2 - \frac{2}{z}(1 - z)^2 \log(1 - z)\right| = \left|\sum_1^\infty \frac{4z^{n+1}}{n(n+1)(n+2)}\right|$$

$$< |z|^2 < |z| \qquad (0 < |z| < 1).$$

The inequality in (2.9) follows from the identity

$$4\sum_1^\infty \frac{1}{n(n+1)(n+2)} = 1.$$

It is also obvious that (2.8) is satisfied for $z = 0$. This completes the proof that $S(z)$ is univalent and starlike with respect to the origin for $|z| < 1$. It is well known that $K(z)$ is also starlike in E; moreover, $\frac{1}{2}S(z)$ is its average value on the interval $[0, z]$. We have proved the following proposition.

LEMMA 1. *Let* $K(z)$ *denote the Koebe function* $z(1 - z)^{-2}$, *which is univalent and starlike with respect to the origin for* $|z| < 1$. *Then the function*

$$S(z) = \frac{2}{z}\int_0^z K(t)\,dt = \frac{2}{1 - z} + \frac{2}{z}\log(1 - z) = \sum_1^\infty \frac{2n}{n+1} z^n$$

is also univalent and starlike with respect to the origin for $|z| < 1$.

We give a second lemma, closely related to Lemma 1 and with a similar proof. It is of some interest in itself, but we state it here for future reference. It may be of value in the proofs by variational methods of certain theorems related to Theorem 1. The functions $S(z)$, $K(z)$, and linear combinations of them, arise in this alternate approach.

LEMMA 2. *Let* $S(z)$ *and* $K(z)$ *be defined as in Lemma* 1. *Then*

$$T(z) = [K(z) - S(z)]^{1/2}$$

is univalent and starlike in $|z| < 1$ *and satisfies the inequalities*

$$|T(z)| \leq \frac{1 - |z|}{|1 - z|} T(|z|) < \left|\frac{z}{1 - z}\right| = |z K(z)|^{1/2} \qquad (0 < |z| < 1).$$

Proof. The inequality

$$(2.10)\qquad \left|\frac{z T'(z)}{T(z)} - \frac{1}{2}\right| < \left|\frac{z T'(z)}{T(z)} + \frac{1}{2}\right| \qquad (|z| < 1)$$

is equivalent, for $0 < |z| < 1$, to the relation

$$|4z^2 - 5z + 2 + \frac{2}{z}(1 - z)^3 \log(1 - z)| = \left|\frac{z^2}{3} + \sum_1^\infty \frac{12 z^{n+2}}{n(n+1)(n+2)(n+3)}\right| < |z|^2,$$

in which the inequality follows from the identity

$$\sum_{1}^{\infty} 12\,[n(n+1)(n+2)(n+3)]^{-1} = 2/3\,.$$

From (2.10) it follows that $T(z)$ is starlike with respect to the origin in $|z| < 1$. Since $T'(0) \neq 0$ and $T(0) = 0$, we conclude that $T(z)$ is univalent and starlike. For $0 < |z| < 1$, (2.9) implies that

$$\left|\frac{3z-2}{(1-z)^2} - \frac{2}{z}\log(1-z)\right| < \left|\frac{z}{1-z}\right|^2 = |z\,K(z)|\,,$$

or

$$|T(z)|^2 = |K(z) - S(z)| < |z\,K(z)| \qquad (0 < |z| < 1)\,.$$

Replacing z by $|z|$ in (2.9), we also readily see that

$$|T(z)| \leq \frac{1-|z|}{|1-z|}\,T(|z|) < \left|\frac{z}{1-z}\right| \qquad (0 < |z| < 1)\,.$$

3. PROOFS OF THE THEOREMS

Let $\mu_0(z)$ be defined as in (2.3). Then

$$\mu_0(z) = \frac{1}{z}\int_0^z \frac{1+t}{1-t}\,dt = \int_0^1 \frac{1+z\rho}{1-z\rho}\,d\rho = -\frac{2}{z}\log(1-z) - 1\,.$$

R. M. Robinson [6] has pointed out that $\mu_0(z)$ is convex in E. However, this is also an immediate consequence of Lemma 1. Since $S(z)$ is starlike in E and since

$$z\mu_0'(z) = S(z) = \frac{2}{1-z} + \frac{2}{z}\log(1-z)\,,$$

it follows that $w = \mu_0(z)$ maps E onto a convex domain D_0. This domain is symmetric about the real axis. We shall show that D_0 lies in the half-plane $\Re w > 2\log 2 - 1 > 0$ between the two straight lines $\Im w = \pm\pi$, a fact that is not immediately obvious. We let

$$\mu_0(e^{i\theta}) = U_0(\theta) + i\,V_0(\theta)\,.$$

Then, for $0 < \theta \leq \pi$,

$$U_0(\theta) = -(\cos\theta)\log\{2(1-\cos\theta)\} - 1 + (\pi-\theta)\sin\theta\,,$$

$$V_0(\theta) = (\sin\theta)\log\{2(1-\cos\theta)\} + (\pi-\theta)\cos\theta\,.$$

(3.1) $$\frac{d}{d\theta}V_0(\theta) = -U_0(\theta)\,.$$

From (2.4) and (2.6) we see that $U_0(\theta) = -U(\theta) > 0$. From (3.1) it follows that $V_0'(\theta) < 0$ for $0 < \theta \leq \pi$. Since $V_0(0) = \pi$ and $V_0'(\theta) < 0$ for $0 < \theta \leq \pi$, it now follows that

$$|\Im \mu_0(z)| < \pi \qquad (|z| < 1).$$

Since $U_0(\pi) = 2\log 2 - 1$ and $U_0(0) = +\infty$, and because $\mu_0(z)$ is univalent, convex and real on the real axis,

$$\Re \mu_0(z) > 2\log 2 - 1 \qquad (|z| < 1).$$

Since $P(z) \in \mathfrak{P}$ has a Herglotz representation

$$P(z) = \int_0^{2\pi} \frac{1 + ze^{i\phi}}{1 - ze^{i\phi}}\, d\alpha(\phi) \qquad \left(\int_0^{2\pi} d\alpha(\phi) = 1\right),$$

where $\alpha(\phi)$ is a nondecreasing function of ϕ in $[0, 2\pi]$, we can write

$$\mu(z) = \frac{1}{z}\int_0^z P(t)\,dt = \int_0^{2\pi}\left(\frac{1}{z}\int_0^z \frac{1 + te^{i\phi}}{1 - te^{i\phi}}\,dt\right) d\alpha(\phi)$$

$$= \int_0^{2\pi}\left(\int_0^1 \frac{1 + \rho z e^{i\phi}}{1 - \rho z e^{i\phi}}\,d\rho\right) d\alpha(\phi) = \int_0^{2\pi} \mu_0(ze^{i\phi})\,d\alpha(\phi).$$

We see that if $|z| = r$, then $\mu(z)$ is an average (with a positive weight factor) of values $\mu_0(ze^{i\phi})$, and hence it lies in the convex hull of the values of $\mu_0(z)$ for $|z| = r$. But since $\mu_0(z)$ is univalent and convex, it follows (as R. M. Robinson observed [6]) that $\mu(z)$ is subordinate to $\mu_0(z)$ for $|z| < 1$. Because $\mu(0) = \mu_0(0) = 1$, there exists a bounded function $\omega(z)$, regular in $|z| < 1$, such that

$$\omega(0) = 0, \qquad |\omega(z)| \leq |z| < 1, \qquad \mu(z) = \mu_0\{\omega(z)\}.$$

Hence, if $F(w)$ is an analytic function, regular in the convex domain D_0 that is the image of $|z| < 1$ by the mapping $w = \mu_0(z)$, then

$$F(\mu(z)) = F(\mu_0\{\omega(z)\}).$$

This states that $F(\mu(z))$ is subordinate to $F(\mu_0(z))$ for $|z| < 1$.

Since the value $F(\mu(z))$ for $|z| = r$ always lies within the set of values $F(\mu_0(z))$ for $|z| \leq r$, it follows that

$$\min_{P\in\mathfrak{P}}\ \min_{|z|=r} \Re F(\mu(z)) = \min_{|z|=r} \Re F(\mu_0(z)). \tag{3.2}$$

Thus $P_0(z) = (1 + z)(1 - z)^{-1}$ is an extremal function for the left member of (1.2). If $\varepsilon = e^{i\alpha}$, where α is real, then clearly $P_0(\varepsilon z)$ is also an extremal function, since the right-hand side of (3.2) is unchanged when z is replaced by εz. The only extremal functions are of the form $P_0(\varepsilon z)$, since by Schwarz's Lemma $|\omega(z)| < |z|$ unless $\omega(z) = e^{i\phi}z$. For if $P_1(z)$ is another extremal function, not of the form $P_0(\varepsilon z)$, then

$$\min_{|z|=r} \Re F\left(\frac{1}{z}\int_0^z P_1(t)\,dt\right) = \min_{|z|=r} \Re F(\mu_0(z)).$$

Then there exists an $\omega_1(z)$ ($|\omega_1(z)| \le |z| < 1$), regular in $|z| < 1$, such that

$$\min_{|z|=r} \Re F(\mu_0\{\omega_1(z)\}) = \min_{|z|=r} \Re F(\mu_0(z)),$$

which for nonconstant $F(z)$ is impossible by the minimum modulus theorem, unless $|\omega_1(z)| = |z|$. If $\omega_1(z) = \varepsilon z$, where $|\varepsilon| = 1$, then

$$\frac{1}{z}\int_0^z P_1(t)\,dt = \mu_0(\omega_1(z)) = \frac{1}{\varepsilon z}\int_0^{\varepsilon z} \frac{1+t}{1-t}\,dt = \frac{1}{z}\int_0^z \frac{1+\varepsilon\rho}{1-\varepsilon\rho}\,d\rho.$$

Thus, for all z and z_1 in E, we see that

$$\int_0^z \left(P_1(\rho) - \frac{1+\varepsilon\rho}{1-\varepsilon\rho}\right) d\rho = 0, \qquad \int_0^{z_1} \left(P_1(\rho) - \frac{1+\varepsilon\rho}{1-\varepsilon\rho}\right) d\rho = 0.$$

For $z \neq z_1$, this gives the relation

$$\frac{1}{z - z_1}\int_{z_1}^z \left(P_1(\rho) - \frac{1+\varepsilon\rho}{1-\varepsilon\rho}\right) d\rho = 0.$$

Letting $z_1 \to z$, we conclude that

$$P_1(z) = \frac{1+\varepsilon z}{1-\varepsilon z} = P_0(\varepsilon z).$$

This completes the proof of Theorem 1.

Corollary 1 follows as a consequence of Theorem 1. Since $\Re f'(z) > 0$, we may write

$$f'(z) = (\cos\alpha)P(z) + i\sin\alpha \qquad (\cos\alpha > 0,\ P(z) \in \mathfrak{P}),$$

$$\frac{f(z)}{z} = i\sin\alpha + \frac{\cos\alpha}{z}\int_0^z P(t)\,dt = i\sin\alpha + (\cos\alpha)\mu(z).$$

When

$$P(z) = P_0(\varepsilon z) = \frac{1+\varepsilon z}{1-\varepsilon z} \qquad (|\varepsilon| = 1),$$

then

$$f(z) = f_0(z) = -e^{-i\alpha}z - \frac{2}{\varepsilon}(\cos\alpha)\log(1-\varepsilon z).$$

The function

$$w = w(z) = \frac{\varepsilon}{z} f_0(\bar{\varepsilon} z) = -e^{-i\alpha} - \frac{2}{z}(\cos\alpha)\log(1-z)$$

is convex in $|z| < 1$, since $z w'(z) = (\cos\alpha) S(z)$ is starlike in $|z| < 1$. Corollary 1 now follows immediately from Theorem 1.

Corollary 2 follows from (1.6) if we take

$$F(P(z)) = G((\cos\alpha) P(z) + i \sin\alpha) = G(f'(z)).$$

We shall now deduce Theorem 2 from Theorem 1. Clearly, for $|z_1| = r$ $(0 < r < 1)$,

$$\min_{P \in \mathfrak{P}} \min_{|z|=r} \Re F(\mu(z)) \leq \Re F(\mu(z_1)). \tag{3.3}$$

If the left-hand side of (3.3) is attained at the point z_0 $(|z_0| = r)$ when

$$P(z) = P_0(z) = (1+z)(1-z)^{-1},$$

and if $z_0 z^{-1} = \varepsilon$, then ε depends on F and z_1. Furthermore, equality occurs in (3.3) if on the right-hand side of (3.3) in the definition of

$$\mu(z_1) = \int_0^1 P(\rho z_1)\, d\rho$$

$P(z)$ is replaced by $P_0(\varepsilon z) = P_0(z_0 z_1^{-1} z)$. In this case, the right-hand side becomes $\Re F(\mu_0(z_0))$. However, the left-hand side is also $\Re F(\mu_0(z_0))$. Hence it follows from (3.3) that

$$\min_{P \in \mathfrak{P}} \Re F(\mu(z_1)) \text{ is attained for } P(z) = P_0(\varepsilon z).$$

This completes the proof of Theorem 2.

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