BOUNDARY BEHAVIOR OF NORMAL FUNCTIONS DEFINED IN THE UNIT DISK

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1. INTRODUCTION

In generalizing some results of Plessner, Meier [12, p. 241] showed that if f is holomorphic for $|z| < 1$, and if there exists a set M of points $e^{i\theta}$ such that for each $e^{i\theta} \in M$, f remains bounded on two distinct rectilinear segments terminating in $e^{i\theta}$, then f has an angular limit at almost all points of M.

The search for an analogous result for holomorphic functions which are normal if $|z| < 1$ led to some results which are given in Section 1. (For the definition of a normal function see [10, p. 53].) Section 2 contains some examples of the behavior enunciated in Section 1.

The point $e^{i\theta}$ is called a Plessner point of f, if the cluster set at $e^{i\theta}$ of f on every Stolz domain with vertex $e^{i\theta}$ is the whole complex plane.

In Section 3 it is shown that, at a Plessner point $e^{i\theta}$ of a normal, meromorphic function f, the cluster set along any chord terminating in $e^{i\theta}$ is the whole Riemann sphere. The case in which the range of such functions f is a nowhere dense set of the Riemann sphere is treated in Section 4.

We conclude, in Section 5, by considering points at which the two-segment property of Bagemihl [1, p. 380] holds.

The following notation will be used. The open unit disk $|z| < 1$ and its circumference $|z| = 1$ will be denoted by D and C, respectively. For z_1 and z_2 in D, set $\rho(z_1, z_2)$ equal to the non-Euclidean (hyperbolic) distance between these points. For a discussion of this non-Euclidean geometry see, for example, [7, Chapter 2].

For any point $e^{i\theta}$, we denote by $\Delta_{\alpha,\theta}$ the symmetric Stolz domain of opening 2α $(0 < \alpha < \pi/2)$ with vertex $e^{i\theta}$. If it is clear that the vertex is at $e^{i\theta}$, we write Δ_α for $\Delta_{\alpha,\theta}$; and if there is no need to distinguish the angle α of opening, we just write Δ.

Let Σ be any subset of D whose closure intersects C only at $e^{i\theta}$, and let $\mathscr{R}$ be the Riemann sphere. For a complex-valued function f defined in D, the cluster set (respectively range set) of f along Σ will be denoted by $C_\Sigma(f, e^{i\theta})$ (respectively $R_\Sigma(f, e^{i\theta})$). It is the set of all points w on $\mathscr{R}$, such that there exists a sequence of points $\{z_n\}$ in Σ with $\lim_{n\to\infty} z_n = e^{i\theta}$ and with $\lim_{n\to\infty} f(z_n) = w$ (respectively $f(z_n) = w$, $n = 1, 2, \cdots$). If τ is a simple, continuous curve in D one of whose ends is $e^{i\theta}$ and if Δ is any Stolz domain at $e^{i\theta}$, we set

$$\pi_\Delta(f, e^{i\theta}) = \bigcap_\tau C_\tau(f, e^{i\theta}),$$

where the intersection is taken over all such τ lying entirely in Δ.

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Finally define

$$\tilde{R}_{\triangle}(f, e^{i\theta}) = \bigcap_{\triangle^*} \operatorname{int} R_{\triangle^*}(f, e^{i\theta}),$$

where $\triangle$ is any Stolz domain at $e^{i\theta}$, and the intersection is taken over all Stolz domains $\triangle^*$ at $e^{i\theta}$ which strictly contain $\triangle$. (The set int $R_{\triangle^*}(f, e^{i\theta})$ is the interior of $R_{\triangle^*}(f, e^{i\theta})$ relative to $\mathscr{R}$. We consider the domain of values of f, as well as the sets $C_{\triangle}(f, e^{i\theta})$, $R_{\triangle}(f, e^{i\theta})$, and $\pi_{\triangle}(f, e^{i\theta})$, to be a subset of $\mathscr{R}$, so that, for example, f is considered to be continuous at its poles. Further, $c = \infty$ is an admissible value in any of the theorems unless otherwise specified.)

2. THE BASIC THEOREM

THEOREM 1. *If* f *is normal and meromorphic in* D, *then for any* $e^{i\theta} \in C$ *and any Stolz domain* $\triangle_\alpha$ *at* $e^{i\theta}$,

$$C_{\triangle_\alpha}(f, e^{i\theta}) - \tilde{R}_{\triangle_\alpha}(f, e^{i\theta}) \subseteq \pi_{\triangle_\alpha}(f, e^{i\theta}).$$

Proof. Consider any $c \in C_{\triangle_\alpha}(f, e^{i\theta}) - \tilde{R}_{\triangle_\alpha}(f, e^{i\theta})$. There exists a sequence $\{z_n\}$ ($z_n \in \triangle_\alpha$) with $\lim_{n\to\infty} z_n = e^{i\theta}$ such that

$$\lim_{n\to\infty} f(z_n) = c. \tag{2.0}$$

Let w_n be the unique point on the diameter of C from $e^{i\theta}$ to $-e^{i\theta}$ for which $\rho(z_n, w_n)$ equals the non-Euclidean distance of z_n to this diameter; for any β ($\alpha < \beta < \pi/2$) let H_β denote the connected region bounded by the two hypercycles, symmetric in this diameter, that form the angles β and $-\beta$ with the diameter. There exists some neighborhood U of $e^{i\theta}$ such that, within this neighborhood, $\triangle_\beta$ contains H_β. Further, if we specify β_1 ($\alpha < \beta_1 < \beta$), we may choose a positive integer N so that, for all $n \geq N$,

$$\rho(z_n, w_n) < \frac{1}{2}\log\cot\left(\frac{\pi}{4} - \frac{\beta_1}{2}\right) \equiv M. \tag{2.1}$$

The existence of such an N is insured by observing that since all z_n lie in $\triangle_\alpha$,

$$\overline{\lim_{n\to\infty}}\, \rho(z_n, w_n) \leq \frac{1}{2}\log\cot\left(\frac{\pi}{4} - \frac{\alpha}{2}\right).$$

For convenience we consider all z_n to satisfy (2.1).

As is usual in arguments concerning normal functions, set

$$g_n(\zeta) = f\left(\frac{\zeta + w_n}{1 + \bar{w}_n \zeta}\right) \quad (n = 1, 2, \cdots).$$

By the normality of f, there exists a subsequence $\{g_{n_k}\}$ which converges uniformly in any compact part of $|\zeta| < 1$ to a function g. Next define a sequence $\{\zeta_k\}$ by the equation

(2.2)
$$z_{n_k} = \frac{\zeta_k + w_{n_k}}{1 + \overline{w}_{n_k}\zeta_k}.$$

The inequality (2.1) implies that

$$|\zeta_k| < \frac{e^{2M} - 1}{e^{2M} + 1} \equiv K;$$

and hence if ζ_0 is an accumulation point of $\{\zeta_k\}$, then $|\zeta_0| \leq K$. Let $\{\zeta_p\}$ be a subsequence of $\{\zeta_k\}$ tending to ζ_0. The continuous convergence of $\{g_{n_k}\}$ at ζ_0 together with (2.0) and (2.2) implies that

$$\lim_{p\to\infty} g_{n_p}(\zeta_p) = \lim_{p\to\infty} f(z_{n_p}) = g(\zeta_0) = c;$$

see [8, p. 173 ff.] for a discussion of the notation of continuous convergence.

We now consider two cases, according as to whether $g(\zeta) \equiv c$ or not.

If $g(\zeta) \equiv c$, then f tends uniformly to the value c on the sequence of non-Euclidean disks with centers w_{n_k} and non-Euclidean radius M. Each disk intersects both boundary segments of Δ_α, which in turn implies that $c \in \pi_{\Delta_\alpha}(f, e^{i\theta})$.

Consider the case where $g(\zeta)$ is not identically c. Let D^* denote any fixed non-Euclidean open disk with center ζ_0 which is contained in the open disk

$$|\zeta| < \frac{1}{2}\frac{e^{2M^*} - 1}{e^{2M^*} + 1},$$

where

$$M^* = \frac{1}{2}\log\cot\left(\frac{\pi}{4} - \frac{\beta}{2}\right).$$

This insures that if $\zeta \in D^*$ then, for any positive integer n,

$$x = \frac{\zeta + w_n}{1 + \overline{w}_n\zeta}$$

is contained in H_β.

Let ξ be any point of D^*. Hurwitz's theorem guarantees the existence of an integer k_0 and a sequence of points $\{\xi_k\}$ in D^* which tend to ξ such that

(2.3)
$$g_{n_k}(\xi_k) = g(\xi)$$

for all $k \geq k_0$.

Defining

$$x_k = \frac{\xi_k + w_{n_k}}{1 + \xi_k\overline{w}_{n_k}} \quad (k \geq k_0),$$

and referring to the definition of the family $\{g_n\}$, we find that (2.3) yields the equality $f(x_k) = g(\xi)$. For sufficiently large k, all x_k are contained in $U \cap H_\beta$ and hence in $\triangle_\beta$. It is clear that the point $g(\xi)$ is contained in $R_{\triangle_\beta}(f, e^{i\theta})$. Since g is an open mapping and ξ is an arbitrary point of D^*, $g(D^*)$ is an open set, which contains $g(\zeta_0) = c$ and which is contained in $R_{\triangle_\beta}(f, e^{i\theta})$. The assertion that $g(\zeta)$ is not identically equal to c implies that $c \in \tilde{R}_{\triangle_\alpha}(f, e^{i\theta})$. By our choice of c, this is impossible. Thus $g(\zeta) \equiv c$, and the theorem is proved.

A point $e^{i\theta}$ is called a Fatou point of f(z) if lim f(z) exists uniformly as $z \to e^{i\theta}$ in every Stolz domain with vertex $e^{i\theta}$.

COROLLARY 1. *Let* f *be normal and meromorphic in* D, *and suppose* f *omits the value* c. *Further suppose there exists a subset* E *of* C *of positive linear measure such that for each point* $e^{i\theta} \in E$ *there exists a Jordan curve* τ_θ, *tending to* $e^{i\theta}$ *and lying within some Stolz domain at* $e^{i\theta}$, *such that* $C_{\tau_\theta}(f, e^{i\theta})$ *does not contain* c. *Then almost all points of* E *are Fatou points.*

Proof. Consider any $e^{i\theta} \in E$. For some Stolz domain $\triangle$ at $e^{i\theta}$, $c \notin \pi_\triangle(f, e^{i\theta})$; and since, by hypothesis, $c \notin \tilde{R}_\triangle(f, e^{i\theta})$, we conclude that $c \notin C_\triangle(f, e^{i\theta})$. Thus $e^{i\theta}$ cannot be a Plessner point of f. An application of Plessner's theorem shows that almost all points of E must be Fatou points.

If we restrict f to be holomorphic and normal in D and if we set $c = \infty$ in Corollary 1, we obtain the more special result below.

COROLLARY 2. *Let* f *be holomorphic and normal in* D. *Suppose that there exists a subset* E *of* C *such that, for each point* $e^{i\theta}$ *in* E, f *is bounded on a Jordan curve terminating in* $e^{i\theta}$ *and lying in some Stolz domain at* $e^{i\theta}$. *Then* f *has Fatou points at almost all points of* E.

3. EXAMPLES

The non-tangential approach to $e^{i\theta}$ hypothesized in Theorem 1 is essential. To see this, we define $C(f, e^{i\theta})$, $R(f, e^{i\theta})$, and $\pi(f, e^{i\theta})$ as we did $C_\triangle(f, e^{i\theta})$, $R_\triangle(f, e^{i\theta})$, and $\pi_\triangle(f, e^{i\theta})$, except that we omit the restriction that the sequence $\{z_n\}$, or the curve τ, necessarily approach $e^{i\theta}$ within some Stolz domain. It is not true that for each normal and meromorphic function f

$$C(f, e^{i\theta}) - R(f, e^{i\theta}) \subseteq \pi(f, e^{i\theta}).$$

If u denotes the elliptic modular function defined in D, then $C(u, e^{i\theta})$, for all but a countable set of points S of C, is the whole Riemann sphere, while for $e^{i\theta} \in S$, $C(u, e^{i\theta})$ is either 0, 1, or ∞. For each $e^{i\theta} \in C$ it is known that $\pi(u, e^{i\theta})$ is the empty set [2, Theorem 3, p. 30]. However, for each $e^{i\theta}$, $C(u, e^{i\theta}) - R(u, e^{i\theta})$ contains at least one of the values 0, 1, or ∞.

The notion of normalcy is also necessary in Theorem 1. Seidel and Bagemihl [4, Corollary 5, p. 191] constructed a function f, holomorphic in D, for which the cluster set on almost all radii is the unit circle. According to Plessner's theorem, almost all points of C are then Plessner points of f. Choose a Plessner point $e^{i\theta}$ of f for which the corresponding cluster set is the unit circle. For any Stolz domain $\triangle$ at $e^{i\theta}$, the point $c = \infty$ is contained in $C_\triangle(f, e^{i\theta}) - \tilde{R}_\triangle(f, e^{i\theta})$ but is not contained in the set $\pi_\triangle(f, e^{i\theta})$.

4. BEHAVIOR OF NORMAL FUNCTIONS AT PLESSNER POINTS

In the sequel let $\triangle$ represent any symmetric or non-symmetric Stolz domain at $e^{i\theta}$. Following the notation used by Noshiro and others [13, p. 68 ff.], if f is meromorphic in D, we denote by J(f) the set of all points $e^{i\theta}$ such that for every Stolz domain at $e^{i\theta}$,

$$C_\triangle(f, e^{i\theta}) = C(f, e^{i\theta}).$$

Let K(f) denote the set of all points $e^{i\theta}$ such that for any two Stolz domains $\triangle_1$ and $\triangle_2$ at $e^{i\theta}$,

$$C_{\triangle_1}(f, e^{i\theta}) = C_{\triangle_2}(f, e^{i\theta}).$$

Also, given any $e^{i\theta}$, let $\rho(\alpha)$ denote that chord of C terminating at $e^{i\theta}$ and making the angle α $(-\pi/2 < \alpha < +\pi/2)$ with the radius at $e^{i\theta}$.

If $\triangle$ is any Stolz domain at $e^{i\theta}$, set

$$\hat{C}_\triangle(f, e^{i\theta}) = \bigcap_{\triangle^*} C_{\triangle^*}(f, e^{i\theta}),$$

where the intersection is taken over all $\triangle^*$ strictly containing $\triangle$. Similarly define, for any chord $\rho(\alpha)$ at $e^{i\theta}$,

$$\hat{C}_{\rho(\alpha)}(f, e^{i\theta}) = \bigcap_{\triangle^*} C_{\triangle^*}(f, e^{i\theta}),$$

where $\triangle^*$ contains $\rho(\alpha)$.

LEMMA 1. *If* f *is normal and meromorphic in* D, *then for any* $e^{i\theta}$ *and any Stolz domain at* $e^{i\theta}$

(i) $$\hat{C}_\triangle(f, e^{i\theta}) = C_\triangle(f, e^{i\theta}).$$

Also, if $\rho(\alpha)$ *is any chord at* $e^{i\theta}$,

(ii) $$\hat{C}_{\rho(\alpha)}(f, e^{i\theta}) = C_{\rho(\alpha)}(f, e^{i\theta}).$$

Proof. We first prove (ii). Consider any fixed $e^{i\theta}$. We need only show that

$$\hat{C}_{\rho(\alpha)}(f, e^{i\theta}) \subseteq C_{\rho(\alpha)}(f, e^{i\theta}).$$

To that end, let $c \in \hat{C}_{\rho(\alpha)}(f, e^{i\theta})$; and let $\{\triangle_n\}$ be a sequence of Stolz domains at $e^{i\theta}$ containing $\rho(\alpha)$, with $\triangle_n \supset \triangle_{n+1}$ and $\bigcap_{n=1}^{\infty} \triangle_n = \rho(\alpha)$. For each $\triangle_n$ $(n = 1, 2, \cdots)$, let $\{z_k^{(n)}\}$ be a sequence contained in $\triangle_n$ such that $z_k^{(n)} \to e^{i\theta}$ and $f(z_k^{(n)}) \to c$. Select a sequence $w_n = z_{k_n}^{(n)}$ $(n = 1, 2, \cdots)$ so that $|w_n - e^{i\theta}| < 1/n$ and $|f(w_n) - c| < 1/n$. Then $f(w_n)$ tends to c as $n \to \infty$, and $w_n \in \triangle_n$. As is easily seen, the non-Euclidean distance of w_n to $\rho(\alpha)$ tends to zero as $n \to \infty$. Let ζ_n be that point on $\rho(\alpha)$ at which this distance is assumed. Then $\rho(w_n, \zeta_n) \to \infty$. By a result of Seidel and Bagemihl [6, Lemma 1, p. 10], $f(\zeta_n) \to c$ as $n \to \infty$. Since c was chosen to be any point contained in $C_{\rho(\alpha)}(f, e^{i\theta})$, (ii) is proved. The proof of (i) may be seen to be similar. With this lemma we can prove the following theorem.

THEOREM 2. *Let* f *be normal and meromorphic in* C. *If* $e^{i\theta} \in K(f)$, *then for any Stolz domain* $\triangle$ *and any chord* $\rho(\alpha)$ *at* $e^{i\theta}$,

$$C_{\triangle}(f, e^{i\theta}) = C_{\rho(\alpha)}(f, e^{i\theta}).$$

Proof. For any two Stolz domains $\triangle_1$ and $\triangle_2$ at $e^{i\theta}$, $C_{\triangle_1}(f, e^{i\theta}) = C_{\triangle_2}(f, e^{i\theta})$. Therefore, employing Lemma 1, we see that

$$C_{\triangle}(f, e^{i\theta}) = \hat{C}_{\rho(\alpha)}(f, e^{i\theta}) = C_{\rho(\alpha)}(f, e^{i\theta}).$$

Remark 1. The set of points K(f), since it contains all Plessner and Fatou points, is of measure 2π on C; and further, according to Collingwood [9, Theorem 4A, p. 383], K(f) is also residual on C.

Remark 2. At any Plessner point $e^{i\theta}$ of f, since $C_{\triangle}(f, e^{i\theta})$ is the Riemann sphere $\mathscr{R}$, so also is the cluster set along any chord. This compares with the behavior of an arbitrary meromorphic function g at a Plessner point, where the equation

$$C_{\rho(\alpha)}(f, e^{i\theta}) = C_{\triangle}(f, e^{i\theta}) = \mathscr{R}$$

is satisfied for a set S of points α $(-\pi/2 < \alpha < \pi/2)$ which is residual relative to the interval $(-\pi/2, \pi/2)$; see Seidel and Bagemihl [3, Theorem 9, p. 1072]. However S may be of measure 0 on $(-\pi/2, \pi/2)$, as an example of Seidel and Bagemihl [5, Corollary 1, p. 82] demonstrates. They constructed a meromorphic and nonconstant function g in D such that for every θ $(0 \leq \theta \leq 2\pi)$ g tends to 0 along almost all chords of C terminating at $e^{i\theta}$. The set of Fatou points must be of measure 0, otherwise, by the uniqueness theorem of Lusin and Privaloff, [11, p. 164] g would be identically 0. By Plessner's theorem, almost all points of C are Plessner points of g. At a Plessner point of g the residual set S, introduced above, must be of measure 0 on $(-\pi/2, \pi/2)$.

5. RESULTS ON THE OUTER ANGULAR CLUSTER SET

If f is holomorphic in D, the outer angular cluster set at $e^{i\theta}$ is defined to be

$$C_A(f, e^{i\theta}) = \bigcup_{\triangle} C_{\triangle}(f, e^{i\theta}),$$

where the union is taken over all Stolz domains at $e^{i\theta}$; for example, see [13, p. 69]. We also form the set

$$\pi_A(f, e^{i\theta}) = \bigcap_{\triangle} \pi_{\triangle}(f, e^{i\theta}),$$

where $\triangle$ varies over all Stolz domains at $e^{i\theta}$.

Finally let C(f) (respectively R(f)) denote, as usual, the set of all values w in $\mathscr{R}$ such that there exists a sequence $\{z_n\}$ in D for which $\lim_{n\to\infty} |z_n| = 1$ and $\lim_{n\to\infty} f(z_n) = w$ $(f(z_n) = w)$.

Let $\mathscr{F}C(f)$ denote the frontier of $C(f)$ relative to $\mathscr{R}$. Collingwood [9, Theorem 9A, p. 389] showed that for any f meromorphic in D and such that $C(f) = \mathscr{F}C(f)$, then for each $e^{i\theta} \in J(f)$,

$$C(f, e^{i\theta}) = \pi(f, e^{i\theta}).$$

An analogous result is given by the next theorem.

THEOREM 3. *Let* f *be meromorphic in* D, *and suppose* $R(f) = \mathscr{F}R(f)$. *Then*

$$C_A(f, e^{i\theta}) = \pi_A(f, e^{i\theta})$$

for every $e^{i\theta} \in C$.

Proof. The hypotheses imply that there exist three values that f takes on at most a finite number of times in D; hence as can be seen from a result of Lehto and Virtanen [10, p. 54], f is normal in D. Further, since $\operatorname{int} R_\triangle(f, e^{i\theta}) = \emptyset$ for any $e^{i\theta}$ and any Stolz domain at $e^{i\theta}$, Theorem 1 yields the conclusion

$$C_\triangle(f, e^{i\theta}) = \pi_\triangle(f, e^{i\theta}), \tag{5.0}$$

where $\triangle$ is any symmetric Stolz domain at $e^{i\theta}$. If $\triangle_1$ and $\triangle_2$ are two arbitrary Stolz domains at $e^{i\theta}$, let $\triangle_3$ be any symmetric Stolz domain at $e^{i\theta}$ which contains both $\triangle_1$ and $\triangle_2$. Appealing to the properties of the sets involved and utilizing (5.0), we conclude that

$$C_{\triangle_1}(f, e^{i\theta}) = \pi_{\triangle_2}(f, e^{i\theta}),$$

and the theorem is proved.

COROLLARY 3. *If* f *is meromorphic in* D *and* $R(f) = \mathscr{F}R(f)$, *then for any* $e^{i\theta} \in J(f)$,

$$C(f, e^{i\theta}) = \pi_A(f, e^{i\theta}).$$

Proof. For $e^{i\theta} \in J(f)$, $C_\triangle(f, e^{i\theta}) = C(f, e^{i\theta})$. An application of Theorem 3 completes the proof.

6. NORMAL FUNCTIONS HAVING THE TWO SEGMENT PROPERTY OF BAGEMIHL

For a given complex-valued function w defined in D suppose that there exist two Jordan curves Γ_1 and Γ_2, lying in D and terminating at $e^{i\theta}$, such that

$$C_{\Gamma_1}(w, e^{i\theta}) \cap C_{\Gamma_2}(w, e^{i\theta}) = \emptyset.$$

In this instance w is said to possess the two-arc property at $e^{i\theta}$. If Γ_1 and Γ_2 can be taken as chords, w is said to possess the two-segment property at $e^{i\theta}$. (For these definitions, see [2, p. 29].)

We find it useful to define a slightly different property. The function w is said to possess the non-tangential two-arc property if there exist two Jordan curves Γ_1 and Γ_2, terminating at $e^{i\theta}$ and lying in some Stolz domain at $e^{i\theta}$, such that

$$C_{\Gamma_1}(w, e^{i\theta}) \cap C_{\Gamma_2}(w, e^{i\theta}) = \emptyset .$$

In the above definitions, if we replace the number 2 by any positive integer n, we alter the definitions accordingly.

THEOREM 5. *Let* f *be normal and meromorphic in* D. *For any positive integer* n, *the set of all points* $e^{i\theta}$ *at which* f *possesses the* n-*segment property is a set of first category and measure* 0 *on* C.

Proof. Consider any fixed positive integer n, and let S(f) denote the set of all $e^{i\theta}$ at which f possesses the n-segment property. We shall show that

$$S(f) \cap K(f) = \emptyset .$$

Since K(f) is known to be a residual set of measure 2π on C, the theorem will then be proved. Suppose there exists a point $e^{i\theta}$ such that $e^{i\theta} \in S(f) \cap K(f)$. Let $\triangle$ be any fixed Stolz domain at $e^{i\theta}$ which contains the n-segments whose corresponding cluster sets have empty intersection. By Theorem 2 the cluster set along any chord, lying in $\triangle$ and terminating at a point of K(f), is $C_\triangle(f, e^{i\theta})$. This contradicts the hypothesis; hence the theorem is proved.

THEOREM 6. *Suppose* f *is meromorphic in* D *and that there exist three values which* R(f) *omits. Then, for each positive integer* n, *the set of all points* $e^{i\theta}$ *at which* f *possesses the non-tangential* n-*arc property is a subset of* C *of first category and linear measure* 0.

Proof. Let n be a fixed positive integer. Since there exist three values of $\mathscr{R}$ which f takes on only a finite number of times, f is normal in D. Let A(f) denote the set of all those points of C at which f possesses the non-tangential n-arc property. As in the previous theorem we show that $A(f) \cap K(f) = \emptyset$. Suppose $e^{i\theta} \in A(f) \cap K(f)$. Set $\triangle$ equal to any fixed symmetric Stolz domain at $e^{i\theta}$ which contains the n Jordan curves whose corresponding cluster sets have empty intersection. By Theorem 1,

$$C_\triangle(f, e^{i\theta}) - \tilde{R}_\triangle(f, e^{i\theta}) = \emptyset .$$

Let $\triangle_1$ be any Stolz domain at $e^{i\theta}$ that strictly contains $\triangle$. Observing that

$$C_{\triangle_1}(f, e^{i\theta}) \supseteq \operatorname{int} R_{\triangle_1}(f, e^{i\theta}) \supseteq \tilde{R}_\triangle(f, e^{i\theta})$$

and since

$$e^{i\theta} \in K(f) \quad \text{and} \quad C_{\triangle_1}(f, e^{i\theta}) = C_\triangle(f, e^{i\theta}),$$

we conclude that

$$C_{\triangle_1}(f, e^{i\theta}) - \operatorname{int} R_{\triangle_1}(f, e^{i\theta}) = \emptyset . \tag{6.0}$$

Consequently, $C_{\triangle_1}(f, e^{i\theta})$ is a nonempty, open and closed, connected subset of $\mathscr{R}$, which implies

$$C_{\triangle_1}(f, e^{i\theta}) = \mathscr{R} . \tag{6.1}$$

By the hypothesis, int $R_{\Delta_1}(f, e^{i\theta})$ must omit at least three values. Combining (6.0) and (6.1), we reach a contradiction, and hence $A(f) \cap K(f) = \emptyset$.

Theorem 6 can be slightly revised.

THEOREM 7. *If* f *is meromorphic and normal in* D *and* R(f) *omits at least one value, then the set of all points of* C *at which* f *has the non-tangential* n-*arc property is a set of first category and linear measure* 0 *relative to* C.

Proof. If the normality of f is assumed, the arguments used to prove Theorem 6 depend only on the fact that int R(f) is not the whole Riemann sphere. Hence the same arguments prove Theorem 7.

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