INDECOMPOSABLE REPRESENTATIONS OF NON-CYCLIC GROUPS

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INTRODUCTION

Let ZG denote the group ring of a finite group G over the ring of rational integers Z. The term "ZG-module" denotes here a left ZG-module which has a finite Z-basis. (A corresponding definition holds when Z is replaced by another ground ring.) Heller and Reiner [2] recently proved

(I) *If* G *has a non-cyclic* p-*Sylow subgroup for some* p, *then there exist infinitely many non-isomorphic indecomposable* ZG-*modules.*

The crux of the proof of (I) was the following result:

(II) *Let* H *be a direct product of two cyclic groups, each of prime order* p. *Then there exist infinitely many non-isomorphic indecomposable* ZH-*modules.*

This latter result was established in [2] by the explicit construction of such indecomposables for $p = 2$, whereas for $p > 2$ a non-constructive proof was given which used the non-periodicity of the homology sequence for H. In the present note, a constructive proof is given for all p. This construction is of especial interest because of the paucity of examples of indecomposable modules.

1. CONSTRUCTION OF THE MODULES

Let p be a fixed prime, and let $H = [a] \times [b]$ be the direct product of two cyclic groups [a] and [b], with generators a and b, respectively, each of order p. Corresponding to each positive integer n, we shall here construct a ZH-module M_n, postponing until Section 2 the proof that each M_n is indecomposable. In order to avoid writing $(-1)^n$ repeatedly, we shall treat only the case where n is even. The reader should have no difficulty in carrying out the analogous construction for odd n.

The following symbols shall constitute a Z-basis for M_n:

$$(1)\qquad \begin{cases} x_1, ax_1, \cdots, a^{p-2}x_1; x_2, bx_2, \cdots, b^{p-2}x_2; \cdots, \\ \quad x_{n-1}, ax_{n-1}, \cdots, a^{p-2}x_{n-1}; x_n, bx_n, \cdots, b^{p-2}x_n; \\ y_1; y_2, ay_2, \cdots, a^{p-2}y_2; \cdots, y_{n-1}; y_n, ay_n, \cdots, a^{p-2}y_n; y_{n+1}\,. \end{cases}$$

In order to define the action of H on M_n, it suffices to define the action of the generators a and b on each of the above basis elements, and then to extend the action of a and b by linearity. Of course, we shall have to verify that

$$(a^p - 1)m = (b^p - 1)m = 0, \ abm = bam \qquad (m \in M)\,.$$

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Let

$$\phi(t) = t^{p-1} + t^{p-2} + \cdots + t + 1$$

be the cyclotomic polynomial of order p. We now define

$$\text{(2)} \quad \left.\begin{array}{ll} a(a^i x_j) = a^{i+1} x_j, & a(a^i y_k) = a^{i+1} y_k, \\ b(b^i x_k) = b^{i+1} x_k, & b(b^i y_k) = b^{i+1} y_k \end{array}\right\} \quad (0 \leq i \leq p - 3, \text{ j odd, k even}),$$

$$\text{(3)} \quad \left.\begin{array}{lll} \phi(a)x_j = y_j, & (a - 1)y_j = 0, & \phi(a)y_k = 0, \\ \phi(b)x_k = y_{k+1}, & (b - 1)y_{k+1} = 0, & \phi(b)y_k = 0 \end{array}\right\} \quad (\text{j odd, k even}),$$

$$\text{(4)} \quad (b - 1)a^i x_j = a^i y_{j+1} = b^i y_{j+1} = (a - 1)b^i x_{j+1} \quad (\text{j odd, } 0 \leq i \leq p - 2).$$

Associativity is easily verified. Next, we have

$$(a^p - 1)x_j = (a - 1)\phi(a) x_j = (a - 1)y_j = 0 \qquad (\text{j odd}),$$

$$(a^p - 1)x_k = \phi(a)(a - 1)x_k = \phi(a) y_k = 0 \qquad (\text{k even}),$$

whence also $(a^p - 1)a^i x_j = 0$, and so on. Thus M_n is annihilated by $a^p - 1$, and similarly by $b^p - 1$. We must also verify that

$$\text{(5)} \qquad ab \cdot m = ba \cdot m \qquad (m \in M).$$

This is obvious when m is any y_i. On the other hand, for odd j and $0 \leq i \leq p - 3$ we have

$$\{(b - 1)a\} a^i x_j = a^{i+1} y_j = \{a(b - 1)\} a^i x_j .$$

Thus (5) holds when $m = a^i x_j$ (j odd, $0 \leq i \leq p - 3$). To prove (5) for $m = a^{p-2}x_j$ (j odd), it suffices to remark that

$$\{(b - 1)\phi(a)\} x_j = 0 = \{\phi(a)(b - 1)\} x_j \qquad (\text{j odd}).$$

Similarly one verifies (5) for $m = b^i x_k$ ($0 \leq i \leq p - 2$, k even). Thus M_n is a ZH-module. Since the Z-rank (M_n: Z) increases steadily with n, it is clear that $M_n \cong M_q$ only when n = q.

2. PROOF OF INDECOMPOSABILITY

Keeping the previous notation, we now show that each module M_n is indecomposable. Setting $\bar{Z} = Z/pZ$ (a field with p elements), $\bar{M}_n = M_n/pM_n$, we see that $\bar{M}_n$ is a $\bar{Z}G$-module. We shall prove that $\bar{M}_n$ is indecomposable, whence M_n is also indecomposable.

In $\bar{Z}G$, we set $A = a - 1$, $B = b - 1$. Then we have

$$A^p = B^p = 0, \qquad AB = BA,$$

and also

$$\phi(a) = A^{p-1}, \qquad \phi(b) = B^{p-1}.$$

The module $\bar{M}_n$ has the following symbols as $\bar{Z}$-basis (we should write $\bar{x}_1, \bar{y}_1, \cdots,$ to indicate that these elements lie in $\bar{M}_n$ rather than in M_n; but for convenience we omit the bars):

$$(6)\qquad \begin{cases} x_1, Ax_1, \cdots, A^{p-2}x_1; x_2, Bx_2, \cdots, B^{p-2}x_2; \cdots, \\ \quad x_{n-1}, Ax_{n-1}, \cdots, A^{p-2}x_{n-1}; x_n, Bx_n, \cdots, B^{p-2}x_n; \\ y_1; y_2, Ay_2, \cdots, A^{p-2}y_2; \cdots, y_{n-1}; y_n, Ay_n, \cdots, A^{p-2}y_n; y_{n+1}. \end{cases}$$

The action of H on $\bar{M}_n$ can be given by specifying how A and B act on the above basis elements. The analogues of formulas (2) are valid, with A and B in place of a and b, respectively. Moreover,

$$(7)\qquad \left.\begin{array}{l} A^{p-1}x_j = y_j, \quad Ay_j = 0, \quad A^{p-1}y_k = 0, \\ B^{p-1}x_k = y_{k+1}, \quad By_{k+1} = 0, \quad B^{p-1}y_k = 0 \end{array}\right\} \text{ (j odd, k even)},$$

$$(8)\qquad BA^i x_j = A^i y_{j+1} = B^i y_{j+1} = AB^i x_{j+1} \qquad (\text{j odd, } 0 \leq i \leq p-2).$$

We find at once that

$$(9)\qquad \begin{cases} A^{p-1}x_1 = y_1, \; A^{p-1}x_2 = A^{p-2}y_2, \; \cdots, \; A^{p-1}x_{n-1} = y_{n-1}, \; A^{p-1}x_n = A^{p-2}y_n, \\ B^{p-1}x_1 = A^{p-2}y_2, \; B^{p-1}x_2 = y_3, \; \cdots, \; B^{p-1}x_{n-1} = A^{p-2}y_n, \; B^{p-1}x_n = y_{n+1}, \end{cases}$$

and that both A^{p-1} and B^{p-1} annihilate all other basis elements listed in (6). Let Y be the $\bar{Z}$-space spanned by

$$\{y_1, A^{p-2}y_2, \cdots, y_{n-1}, A^{p-2}y_n, y_{n+1}\},$$

let $X = \Sigma_1^n \bar{Z}x_i$, and let $\theta: \bar{M}_n \to X$ be the projection of $\bar{M}_n$ onto its $\bar{Z}$-subspace X. It follows at once from (9) that if N is a $\bar{Z}H$-submodule of M_n such that

$$(10)\qquad (\theta(N) : \bar{Z}) = r,$$

where $r > 0$, then

$$(A^{p-1}N + B^{p-1}N : \bar{Z}) \geq r + 1.$$

However,

$$A^{p-1}N + B^{p-1}N \subset N \cap Y,$$

so that

$$(11)\qquad (N \cap Y : \bar{Z}) \geq r + 1.$$

On the other hand, any non-zero $\bar{Z}H$-submodule L of $\bar{M}_n$ must satisfy

$$(L \cap Y : \bar{Z}) > 0,$$

as is easily deduced from the fact that for each i, $A^i L \subset L$ and $B^i L \subset L$. Therefore (10) implies (11) even when $r = 0$, provided that $N \neq 0$.

We are now ready to show that $\bar{M}_n$ is indecomposable. For suppose that

$$M_n = N_1 \oplus N_2 \tag{12}$$

is a decomposition into non-zero $\bar{Z}H$-submodules, and let $n_i = (\theta(N_i) : \bar{Z})$ $(i = 1, 2)$. Then

$$(\theta(N_1 + N_2) : \bar{Z}) = (\theta(M_n) : \bar{Z}) = n,$$

which shows that $n_1 + n_2 \geq n$. By the preceding remarks,

$$(N_i \cap Y : \bar{Z}) \geq n_i + 1 \qquad (i = 1, 2).$$

But then $N_1 \cap Y$ and $N_2 \cap Y$ are subspaces of the $(n + 1)$-dimensional space Y, and we have

$$(N_1 \cap Y : \bar{Z}) + (N_2 \cap Y : \bar{Z}) \geq n_1 + 1 + n_2 + 1 > n + 1.$$

Consequently

$$(N_1 \cap Y) \cap (N_2 \cap Y) \neq 0,$$

and so also $N_1 \cap N_2 \neq 0$. This contradicts (12), and proves that $\bar{M}_n$ is indecomposable.

3. INDECOMPOSABLE MODULES FOR ARBITRARY GROUPS

In order to demonstrate the usefulness of the preceding proof, we show briefly how (I) may be deduced from (II). Let G be a finite group having a non-cyclic p-Sylow subgroup S, and let Z^* be the ring of p-adic integers in the p-adic completion of the rational field. Suppose there are finitely many non-isomorphic ZG-modules, say $L_1, \cdots, L_t$. We shall obtain a contradiction.

Since S is non-cyclic, the group H is a homomorphic image of S, and thus each M_n is also an indecomposable ZS-module. Form the induced module

$$M_n^G = ZG \otimes_{ZS} M_n.$$

Then we may write

$$M_n^G = a_1 L_1 \oplus \cdots \oplus a_t L_t \qquad (a_i \in Z,\ a_i \geq 0),$$

indicating thereby that M_n^G splits into a direct sum of a_1 copies of L_1, and so on. If we set $M_n^* = Z^* \otimes_Z M_n$, $L_i^* = Z^* \otimes_Z L_i$, this yields

$$(M_n^*)^G = a_1 L_1^* \oplus \cdots \oplus a_t L_t^*.$$

Now we remark that M_n^* is a direct summand of $(M_n^*)^G$ restricted to S, as follows at once from the definition of induced modules. Furthermore (see Borevich and Faddeev [1], Reiner [3], or Swan [4]), the Krull-Schmidt theorem holds for

Z^*S-modules. Finally, $M_n^*/pM_n^* \cong \bar{M}_n$, and by the results of Section 2 it follows that M_n^* is indecomposable. Therefore we may conclude that M_n^* is (as Z^*S-module) a direct summand of some L_i^* viewed as Z^*S-module. But this places an upper bound on the Z-rank of M_n, and so gives a contradiction. We have thus shown (as in [2]) that (II) implies (I).

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