THE POINCARÉ DUALITY IN GENERALIZED MANIFOLDS

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1. INTRODUCTION

The generalized manifolds, that is, topological spaces having the local homology properties of manifolds, have been studied notably by Čech, Lefschetz, Begle [2, 3], and Wilder [9]; the two last-named authors proved, among other results, a Poincaré duality theorem which is also valid in the noncompact case. The main purpose of this paper is to give a simple proof, within the framework of sheaf theory, of such a theorem. The theorem involves Alexander-Spanier cohomology and Alexander-Spanier cohomology with compact carriers (in the sense of [4], not of [6]; see below), and it is proved in Section 3 under a condition more general than Wilder's, not for the sake of generality, but because this simplifies the exposition. Its relationship to the Begle-Wilder theorem is discussed in Section 7; Sections 4 and 5 introduce local Betti numbers and homological local connectedness; Section 6 is devoted to some results of Wilder which pertain to these notions and are of particular interest for generalized manifolds; the latter are discussed in Section 7.

Notation. All spaces considered here are locally compact (and Hausdorff). $\overline{Y}$ is the closure of a subset Y of the space X; L stands for a principal ideal ring. $C^i(X, L)$ or C^i (resp. $C^i_c(X, L)$ or C^i_c) is the L-module of i-dimensional L-valued Alexander-Spanier cochains in X (resp. with compact carriers) (as defined, for example, in [4a, Exposé VI], under the name of Čech-Alexander cochains of the first (resp. second) kind). $C^*(X, L)$ or C^* (resp. $C^*_c(X, L)$ or C^*_c) is the direct sum of the $C^i(X, L)$ (resp. $C^i_c(X, L)$), endowed with the usual boundary operator raising degrees by one; and $H^*(X, L)$ (resp. $H^*_c(X, L)$) is the resulting cohomology group: the Alexander-Spanier cohomology group (resp. with compact carriers) of X, and with coefficients in L. As is well known, $H^*(X, L)$ may be identified with the Čech cohomology based on infinite coverings, and if $X = Y - F$, with Y compact and F closed in Y, then $H^*_c(X, L)$ may be identified with the relative Čech cohomology group of Y mod F.

By f^* we denote the homomorphism of $H^*(Y, L)$ in $H^*(X, L)$ induced by a continuous map $f: X \to Y$. In case f is the inclusion of a subspace, it will sometimes be convenient to denote by $H^*(X \subset Y, L)$ the image of f^*.

Let U be an *open* subset of X. Then $C^*_c(U, L)$ may be identified with the subgrating of elements in $C^*_c(X, L)$ having carriers in U; and this embedding gives rise to a homomorphism of $H^*_c(U, L)$ in $H^*_c(X, L)$; it will be denoted by j^* or j^*_{UX}, and its image by $H^*_c(U \subset X, L)$. Recall that, given a closed subset F of X, there is an exact cohomology sequence

$$(1) \qquad \cdots \to H^i_c(F, L) \to H^{i+1}_c(X - F, L) \xrightarrow{j^*} H^{i+1}_c(X, L) \to H^{i+1}_c(F, L) \to \cdots.$$

As far as sheaf theory is concerned, we use the terminology of [4b] and assume it to be known. *Grating* will stand for *carapace*, and S(a) will denote the carrier (support) of an element a belonging to a grating A. Given a locally finite covering (U_i) $(i \in I)$, a *partition of unity* for A, subordinate to (U_i), is a family (r_i) $(i \in I)$ of

Received June 26, 1957.

endomorphisms of A (for the L-module structure only) whose sum is the identity, and such that $S(r_i a) \subset S(a) \cap \overline{U}_i$ for all $i \in I$. If this exists for every locally finite covering, A is said to be *fine*. In particular, $C_c^*(X, L)$ is fine.

2. A FINE GRATING

2.1. The duality theorem in Section 3 will be obtained simply by applying the fundamental theorems of sheaf theory to the grating $C_*(X, L)$, where $C_*(X, L)$ is the direct sum of the L-modules $C_i(X, L) = \mathrm{Hom}(C_c^i(X, L), L)$. The boundary operator ∂ of $C_*(X, L)$ shall be the transpose of d; that is,

$$\partial a(c) = a(dc) \qquad (a \in C_i(X, L),\ c \in C_c^{i-1}(X, L))\,;$$

and we shall denote by $H_i(C_*(X, L))$ the corresponding i-homology group.

The carrier $S(a)$ of $a \in C_*(X, L)$ is defined by the rule: the point $x \in X$ is not in $S(a)$ if it has a neighborhood U such that $a(c) = 0$ whenever $S(c) \subset U$. Thus $S(a)$ is closed.

2.2. LEMMA. *Let* X *be paracompact. Then* $C_*(X, L)$, *endowed with the carriers and boundary operator defined above, is a fine grating without torsion, in which locally finite sums converge.*

It follows immediately from the definitions that

$$S(\partial a) \subset S(a), \qquad S(a + a') \subset S(a) \cup S(a'), \qquad S(k \cdot a) \subset S(a)$$

$(a, a' \in C_*(X, L),\ k \in L)$; thus $C_*(X, L)$ is a pregrating: absence of torsion means that $S(k \cdot a) = S(a)$ for $k \in L$ $(k \neq 0)$; this property follows from the fact that L is a domain of integrity.

Let us now show that if $S(a)$ does not meet $S(c)$ $(a \in C_*(X, L),\ c \in C_c^*(X, L))$, then $a(c) = 0$. In fact, $S(c)$ being compact, we can find a finite number of open sets V_i $(1 \leq i \leq k)$ whose union covers $S(c)$, and such that a is zero on any element with support in one of the V_i. Let now (r_j) $(0 \leq j \leq k)$ be a partition of unity for $C_c^*(X, L)$, subordinate to a covering of X formed by the V_i $(1 \leq i \leq k)$, and a V_0 whose closure is in $X - S(c)$. Then $r_0 c = 0$ and $a(r_i c) = 0$ $(1 \leq i \leq k)$; hence

$$a(c) = \sum_0^k a(r_j c) = 0\,.$$

In particular, if $S(a)$ is empty, we have $a = 0$, which means that $C_*(X, L)$ is a grating.

A family (a_i) $(i \in I)$ of elements of $C_*(X, L)$ is said to be *locally finite* if each compact subset of X meets at most a finite number of the $S(a_i)$; in that case, by the above, for any $c \in C_c^*(X, L)$, at most a finite number of the $a_i(c)$ may differ from zero, and their sum is well defined. Thus to the family (a_i) there is assigned a sum $a \in C_*(X, L)$ by the rule $a(c) = \Sigma_{i \in I}\, a_i(c)$, and by definition, this means that the locally finite sums converge in $C_*(X, L)$ [4b, Exp. XVIII, No. 4].

Let now (U_j) $(j \in J)$ be a locally finite open covering of X, and let (r_j) be a partition of unity for $C_c^*(X, L)$, subordinate to (U_j). We define $r_j a$ by $r_j a(c) = a(r_j c)$. Then $S(r_j a) \subset \overline{U}_j \cap S(a)$, and the $r_j a$ form a locally finite family whose sum is clearly a. Thus $C_*(X, L)$ is fine.

2.3. We denote by $\mathscr{F}(X, L)$ (or simply, if it does not lead to confusion, by $\mathscr{F}$) the sheaf associated with the grating $C_*(X, L)$. The stalk $\mathscr{F}_x$ above x is then C_*/C_{*X-x}, where C_{*U} denotes the set of elements in C_* with carriers in the open set U. Then $\mathscr{F}_x$ is a graded, torsion-free L-module, with a boundary operator lowering degrees by 1, induced by ∂. The homology sheaf of $\mathscr{F}$ will be denoted by $H(\mathscr{F})$ or $H(\mathscr{F}(X, L))$. Thus $H(\mathscr{F})_x = H(\mathscr{F}_x)$. The group $H(\mathscr{F}_x)$ is quite analogous to the homology group *in* the point x introduced by Alexandroff [1], and accordingly we may call $H(\mathscr{F})$ the sheaf of local homology groups. In Section 4, we shall discuss these in connection with the Betti numbers *around* the points of X.

For open sets U and V ($U \subset V$) in X, let j_0: $C_*(V, L) \to C_*(U, L)$ be the transposed map of j^0, and let j_* or j_*^{UV}: $H_*(C_*(V, L)) \to H_*(C_*(U, L))$ be the induced map of homology groups. It is readily seen that

$$H(\mathscr{F}_x) = \lim_{\to} (H_*(C_*(U, L)), j_*^{UV}),$$

where U runs through the open neighborhoods of x.

2.4. A grating A is called *complete* provided the natural map of A into the module $\Gamma(\mathscr{F}(A))$ of cross sections of the associated sheaf is an isomorphism. This is the case if A is fine and if the locally finite families of elements in A converge in A [4b, Exp. XVIII, Theorem on p. 9]. Thus, by (2.2), $C_*(X, L)$ is complete, (or, more precisely, is Φ-complete with respect to the family Φ of closed subsets of X).

3. THE DUALITY THEOREM

Let L be a principal ideal ring, and n a nonnegative integer. We consider the following condition for a space X:

(L-n). X *is locally compact, paracompact, finite-dimensional. For each* $x \in X$, *we have* $H_n(\mathscr{F}(X, L)_x) \cong L$, $H_i(\mathscr{F}(X, L)) = 0$ $(i \neq n)$. (The dimension which matters in this paper is the cohomological Φ-dimension with respect to the family Φ of closed sets, as defined in [4b, Exp. XVII, p. 8]. It is majorized by the covering dimension defined by means of locally finite coverings, and minorized by the cohomological dimension introduced by H. Cohen [5].)

A space satisfying (L-n) will be called *locally orientable* (resp. *orientable*) if the sheaf $H_n(\mathscr{F})$ is locally isomorphic (resp. isomorphic), to the constant sheaf $X \times L$. If X satisfies (L-n), so does every open paracompact subspace; if it is moreover orientable or locally orientable, then so is every open paracompact subspace.

3.1. THEOREM. *Let* X *satisfy* (L-n). *Then* $H_{n-i}(C_*(X, L))$ $(i = 0, \pm 1, \pm 2, \cdots)$ *is isomorphic to the* ith *cohomology group* $H^i(X, H_n(\mathscr{F}))$ *of* X *with respect to the sheaf* $H_n(\mathscr{F})$ *of local* n-*dimensional homology groups.*

We change the degrees in $C_*(X, L)$ by writing A^{n-i} instead of $C_i(X, L)$. Then the direct sum A of the A^i is a complete fine grating, with boundary operator raising degrees by 1. Under the assumption (L-n), the homology sheaf $H(\mathscr{F}(A))$ is locally of degree zero and therefore, by [4b, Exp. XIX, Corollary to Theorem 5],

$$H^i(A) \cong H^i(X, H^0(\mathscr{F}(A))),$$

or, in the original notation,

$$H_{n-i}(C_*(X, L)) \cong H^i(X, H_n(\mathscr{F}(X, L))),$$

which is our contention.

3.2. Assume now that L is a field. Then, by the universal coefficient theorem, $H_{n-i}(C_*(X, L))$ is the dual space of $H_c^{n-i}(X, L)$, and the theorem gives

$$H^i(X, H_n(\mathscr{F})) \cong \mathrm{Hom}(H_c^{n-i}(X, L), L) \qquad (i \geq 0).$$

3.3. COROLLARY. *Let* L *be a field, and let* F *be a closed subset whose complement* U *is paracompact.*[1] *Then*

(a) $$H_c^j(U, L) = H_c^j(F, L) = 0 \qquad \text{for } j > n.$$

(b) *Assume moreover that* X *is connected and locally orientable.*

Then $H_c^n(X, L)$ *is* 1-*dimensional or* 0-*dimensional according to whether* X *is orientable or not.* $H_c^n(F, L) = 0$ *if* $F \neq X$.

(a). The left-hand side of (3.2) is of course zero for $i < 0$, whence

$$H_c^j(X, L) = H_c^j(U, L) = 0 \qquad \text{for } j > n;$$

the equality $H_c^j(F, L) = 0$ $(j > n)$ then follows from the exact sequence (1) of Section 1.

(b). $H^0(X, H_n(\mathscr{F}))$ is the module $\Gamma(H_n(\mathscr{F}))$ of cross sections of the sheaf $H_n(\mathscr{F})$; if the latter is locally constant, the set of points where a cross section is zero is open and closed; X being connected, this implies that the dimension of $H^0(X, H_n(\mathscr{F}))$ is either 0 or 1. It follows further that $H_n(\mathscr{F})$ is constant (that is, X is orientable) if and only if $H^0(X, H_n(\mathscr{F}))$ is one-dimensional. This proves the first part of (b).

By (a) and the exact sequence (1), the second statement of (b) is equivalent to the fact that $j^*_{UX}: H_c^n(U, L) \to H_c^n(X, L)$ is surjective or, equivalently, that

$$j_*^{UX}: H_n(C_*(X, L)) \to H_n(C_*(U, L))$$

is injective. But the isomorphism of the theorem, applied to X and U, is compatible with the restriction to U, so that j_*^{UX} may be viewed as the restriction to U of the cross sections of $H_n(\mathscr{F})$ on X; since a nonzero cross section has no zero in our case, this map is indeed injective.

3.4. *Remark on orientability.* $H_n(C_*(X, L)) \cong H^0(X, H_n(\mathscr{F})) \cong \Gamma(H_n(\mathscr{F}))$. Thus our definition of orientability may be phrased in the following way:

(a) X *is orientable if* $C_n(X, L) = \mathrm{Hom}(C_c^n(X, L), L)$ *has a cycle with carrier equal to the whole space.*

If L is a field, and X is separable metric, the proof of 3.3b shows that this is equivalent to

(b) X *is orientable if* $H_c^n(X, L)$ *contains a nonzero element which is in the image of* j^*_{UX} *for every nonempty open subset* U.

Condition (a) is the precise analogue of Wilder's definition: "existence of an infinite cycle which is not carried by a proper closed subset." Condition (b) corresponds to Smith's definition (see Section 7).

3.5. Assume again that L is a field, and that X is locally connected. Then the connected components of X are open, closed, and paracompact. Thus it follows from

1. This will be the case for every open subset if, for example, X is separable metric.

(3.3) that if X is locally orientable, the dimension of $H_c^n(X, L)$ is equal to the number of orientable connected components of X.

3.6. *The cup-product pairing.* The cup-product defines a pairing of $H^i(X, L)$ and $H_c^j(X, L)$ to $H_c^{i+j}(X, L)$; when the latter group is one-dimensional, we may identify it with L and obtain a pairing of $H^i(X, L)$ and $H_c^j(X, L)$ to L. It is said to be orthogonal if in each module the annihilator of the other is reduced to zero. As in the case of ordinary manifolds, we may in the orientable, connected case strengthen (3.2) as follows.

THEOREM. *Suppose that* X *is connected, satisfies* (L-n), *where* L *is a field, and is orientable. Let* ξ *be a nonzero element of* $H^0(X, L) = \mathrm{Hom}(H_c^n(X, L), L)$. *Then the cup-product pairing of* $H^i(X, L)$ *and* $H_c^{n-i}(X, L)$ *to* L *defined by* $(u, v) \to \xi(u \cup v)$ *is orthogonal, and it identifies* $H^i(X, L)$ *with* $\mathrm{Hom}(H_c^{n-i}(X, L), L)$ $(i \geq 0)$.

As in the proof of (3.2), we write A^i for $C_{n-i}(X, L)$; but we use in A the boundary operator ∂' defined by

$$\partial' a = (-1)^{i+1} \partial a \quad (a \in A^i);$$

of course, this does not alter the homology groups. We define a map ϕ: $C^i(X, L) \to A^i$ by

$$\phi(b)(c) = \xi(b \cup c) \qquad (b \in C^i(X, L),\ c \in C_c^{n-i}(X, L)),$$

ξ being identified with a cocycle. The map ϕ is linear and, clearly, the carrier of $\phi(b)$ is contained in S(b). Since ξ, being a cycle, is zero on coboundaries, we have

$$\xi(db \cup c) + (-1)^i \xi(b \cup dc) = 0 \quad (b \in C^i(X, L),\ c \in C_c^{n-i+1}(X, L)),$$

$$\phi(db)(c) = (-1)^{i+1}\phi(b)(dc) = \partial' \phi(b)(c),$$

which means that ϕ commutes with the coboundary operators. It induces then a map ϕ^*: $H^*(C^*(X, L)) \cong H^*(X, L) \to H^*(A)$, and our assertion is clearly equivalent to the proposition that ϕ^* is an isomorphism; we shall now prove this.

The map ϕ defines a map ϕ': $\mathscr{F}(C(X, L)) \to \mathscr{F}(A)$ of the associated sheaves and of their homology sheaves. Both $H^0(\mathscr{F}(C(X, L)) \cong H^0(X, L)$ and $H^0(\mathscr{F}(A))$ are one-dimensional, and since $H^0(X, L)$ is generated by the unit element for the cup-product, ϕ' induces an isomorphism of one onto the other. Since in our case the homology sheaves $H^0(\mathscr{F}(C(X, L))$ and $H^0(\mathscr{F}(A))$ are constant, the map which to each cocycle of $\Gamma\mathscr{F}(C^0(X, L))$ (resp. $\Gamma\mathscr{F}(A^0)$), assigns its value at $x \in X$ induces an isomorphism of $H^0(\mathscr{F}(C^*(X, L))$ onto $H^0(\mathscr{F}(C^*(X, L))_x)$, (resp. $H^0(\mathscr{F}(A))$ onto $H^0(\mathscr{F}(A)_x)$); hence ϕ' is also an isomorphism of $H^0(\mathscr{F}(C^*(X, L))_x)$ onto $H^0(\mathscr{F}(A)_x)$, for all x. It is an isomorphism of $H^i(\mathscr{F}(C^*(X, L))_x)$ onto $H^i(\mathscr{F}(A)_x)$ for $i > 0$, since both groups are then zero. Both $C^*(X, L)$ and A are fine and complete; and the contention that ϕ^* is an isomorphism now follows from the Corollary to Theorem 4 in [4b, Exp. XIX, p. 7].

4. LOCAL BETTI NUMBERS

4.1. Let L be a field. The ith local Betti number $p^i(x)$ or $p^i(x, L)$ of X *around* x may be defined as follows, by means of cohomology [2, 9]: For two open neighborhoods $U \subset V$ of x, let

$$p^i(x, U, V) = \dim H^i_c(U \subset V, L),$$

and let $p^i(x, V)$ be the (possibly transfinite) lower bound of $p^i(x, U, V)$ as U varies inside V. Then $p^i(x)$ is the upper bound of the $p^i(x, V)$ as V runs through a fundamental system of open neighborhoods of x. If $p^i(x)$ is infinite, but the $p^i(x, V)$ are finite, it is said to be *increasingly infinite*. If the $p^i(x, V)$ are infinite for a fundamental system of neighborhoods, $p^i(x)$ is said to be *actually infinite*. Now $H_i(C_*(U, L))$ is the dual space of $H^i_c(U, L)$, and j_*^{UV} is the transpose of j^*_{UV}; since $H_i(\mathscr{F})_x$ is the inductive limit of the $H_i(C_*(U, L))$, we immediately have the following result.

4.2. LEMMA. *If* $p^i(x)$ *is finite and equal to* k, *then so is* $\dim H_i(\mathscr{F})_x$. *If* $\dim H_i(\mathscr{F}_x)$ *is infinite, then so is* $p^i(x)$. *If* $\dim H_i(\mathscr{F})_x$ *is finite, and if* $p^i(x)$ *is at most increasingly infinite, then* $p^i(x)$ *is finite.*

Thus $H_i(\mathscr{F})_x$ bears the same relationship to $p^i(x)$ as the Alexandroff ith homology group *in* x, (see [10]); however, I do not know under what assumptions, if any, these groups are isomorphic. By [10], the Alexandroff group is the inductive limit of the Čech relative homology groups of compact pairs in X - x; the group $H_i(\mathscr{F}_x)$ is here $H_i(C_*/C_{*X-x}, L)$, by definition; when $p^i(x)$ is finite, these spaces have the same dimension and are therefore isomorphic.

4.3. When L is an arbitrary principal ideal ring, we shall, in analogy with the above, use the following definitions: $p^i(x, L)$ is equal to k if corresponding to each open neighborhood U of x there exist open sets $W \subset V \subset U$, containing x and such that for each open neighborhood W' of x in W, $H^i_c(W' \subset V, L)$ is a free L-module with k-generators. If every open neighborhood U of x contains another open neighborhood V of x such that $H^i(V \subset U, L)$ is a finitely generated L-module, then $p^i(x, L)$ is at most increasingly infinite.

5. COHOMOLOGICAL LOCAL CONNECTEDNESS

5.1. In formulating the concept of local connectedness in terms of cohomology, we shall use the symbol clc, in order to avoid conflict with the notation of [9]. In this section, $H^0(X, L)$ is the reduced cohomology group.

The space X is p-clc (over the principal ideal ring L) at x if, given a neighborhood V of x, there exists a neighborhood U of x in V such that $H^p(U \subset V, L) = 0$; it is clc^r at x if it is p-clc at x for all $p \leq r$, and clc at x if this is true for all r. The space X is p-clc, clc^r or clc if it has the corresponding property at every point. Clearly, X is p-clc for all $p > \dim X$. Since X is assumed to be locally compact, we obtain equivalent definitions using only open or closed neighborhoods. We are interested only in finite-dimensional spaces, and thus if X is p-clc at x for all p, then there exists, for a given V, a $U \subset V$ such that $H^p(U \subset V, L) = 0$ simultaneously for all p.

5.2. It is a well-known fact that, given $x \in X$ and $a \in H^p(X, L)$, there exists a neighborhood U of x such that the natural map $H^p(X, L) \to H^p(U, L)$ annihilates a. It follows then that for X to be p-clc at x, it suffices that given a neighborhood U of x, there exists a neighborhood V of x in U such that $H^p(V \subset U, L)$ is a finitely generated L-module.

6. SOME RESULTS OF WILDER

Wilder has drawn interesting consequences from homological local connectedness or finiteness of local Betti numbers; they will be used in Section 7, and we prove them here for the sake of completeness. The proofs are Wilder's, phrased in the technique underlying this paper, and in such a way that they are valid also in the case where the coefficients do not form a field.

All cohomology groups are taken with respect to a fixed principal ideal ring of coefficients L, which will not be mentioned explicitly. "Finitely generated" will always refer to the L-module structure.

6.1. DEFINITION. *The space* X *has property* $(P, Q)_n$ *if, whenever* $\overline{Q} \subset P$ (P, Q *open,* $\overline{Q}$ *compact*), $H_c^n(Q \subset P)$ *is finitely generated.*

This notion was introduced by Wilder [9, p. 193].

6.2. PROPOSITION. *If* X *has property* $(P, Q)_{n+1}$ *and if* $p^n(x)$ *is at most increasingly infinite for all* $x \in X$, *then* X *has property* $(P, Q)_n$.

[9, Chap. VI, Theorem 7.2.] Let P, Q be open, with $\overline{Q}$ compact and contained in P. If R is an open neighborhood of Q in P, then $H_c^n(R \subset P) \supset H_c^n(Q \subset P)$. On the other hand, if the U_i $(1 \leq i \leq k)$ are open sets in P whose union contains Q, then we can find open sets V_i $(1 \leq i \leq k)$ with $\overline{V}_i \subset U_i$, whose union also contains Q. Therefore, using suitable finite coverings of Q, and using induction on the number of elements in the coverings, we can readily see that it is enough to show the following:

Let V_i, U_i be open sets in P such that $\overline{V}_i \subset U_i$ and $\overline{U}_i \subset P$ $(i = 1, 2)$, and let $U = U_1 \cup U_2$, $V = V_1 \cup V_2$. Then, if $H_c^n(U_i \subset P)$ is finitely generated $(i = 1, 2)$, so is $H_c^n(V \subset P)$.

To this end we consider the following commutative diagram

$$\begin{tikzcd}
H_c^n(V_1) + H_c^n(V_2) \arrow[r] \arrow[d, "g_1"] & H_c^n(U_1) + H_c^n(U_2) \arrow[dr, "f_1"] \arrow[d, "g_2"] & \\
H_c^n(V) \arrow[r] \arrow[d, "h_1"] & H_c^n(U) \arrow[r, "f_2"] \arrow[d, "h_2"] & H_c^n(P) . \\
H_c^{n+1}(V_1 \cap V_2) \arrow[r, "k"] & H_c^{n+1}(U_1 \cap U_2) &
\end{tikzcd}$$

The horizontal arrows are the natural inclusion maps j^*, the first two vertical columns are portions of exact Mayer-Vietoris sequences (see Section 8). Let A be the kernel of h_2; by exactness, $f_2(A) = \operatorname{Im} f_1$, and it is finitely generated, since $H_c^n(U_i \subset P)$ has this property by assumption $(i = 1, 2)$. The image of $h_2 \circ j_{VU}^*$ is equal to that of $k \circ h_1$; since Im k is finitely generated by the property $(P, Q)_{n+1}$, we can find a finitely generated submodule B of $H_c^n(U)$ such that $H_c^n(V \subset U) \subset A + B$ (the latter sum not necessarily being direct). Then $H_c^n(V \subset P)$ is contained in $f_2(A) + f_2(B)$, and since both of these are finitely generated, $H_c^n(V \subset P)$ is also finitely generated.

6.3. PROPOSITION. *Let* X *be* clc^n, *and let* P, Q *be subspaces, with* $\overline{P}$ *compact and* $\overline{Q}$ *interior to* P. *Then* $H^n(Q \subset P)$ *is finitely generated.*

[9, Chap. VI, 3.8.] Since Q has an open neighborhood with closure in P, it is enough to show that $H^n(\overline{Q} \subset \overline{P})$ is finitely generated. For a proof of this last statement, see [7, 3.5]; the proof there is given in Čech homology; but the translation into cohomology offers no difficulty.

6.4. THEOREM. *Let* X *be a locally compact, finite-dimensional space. Then* X *is* clc *if and only if, for all* $i \geq 0$, $p^i(x)$ *is at most increasingly infinite for all* $x \in X$.

For the "if" part, see [9, Chap. VI, 7.9]; for the "only if" part, see [9, Chap. VII, 2.25].

(i) We assume first that X is clc, and we have to prove that $p^n(x)$ is at most increasingly infinite for all n and x. If $n > \dim X$, this is clear, and we may therefore assume by induction that our assertion is true for $n + 1$; it will be sufficient to prove the following: Let P be an open relatively compact neighborhood of x, and let U, V, W be open neighborhoods of x such that $\overline{U} \subset P$, $\overline{V} \subset U$, $\overline{W} \subset V$ and $H^n(\overline{U} \subset P) = 0$; then $H^n_c(W \subset V)$ is finitely generated.

We consider the following diagram

$$\begin{tikzcd}
 & & H^n(\overline{U}) & H^n(P) \arrow[l, "k_1"'] \\
H^n_c(W) \arrow[r, "j_1"] \arrow[urr, "\psi"] & H^n_c(V) \arrow[r, "j_2"] & H^n_c(U) \arrow[u, "\alpha_1"'] \arrow[r, "j_3"] & H^n_c(P) \arrow[u, "\alpha_2"'] \\
H^{n-1}(\overline{U} - W) \arrow[u, "\phi"] \arrow[r, "k_2"] & H^{n-1}(\overline{V} - V), \arrow[u, "\phi'"] & &
\end{tikzcd}$$

where the maps j_i, k_i are defined by inclusions, ϕ, ψ (resp. ϕ') are parts of the cohomology sequence of $(\overline{U}, \overline{U} - W)$ (resp. $(\overline{V}, \overline{V} - V)$), α_1 is defined by the inclusion of $C^*_c(U)$ in $C^*(\overline{U})$, and α_2 is defined in the same way as α_1. The commutativity of the diagram is clear. By assumption, $\mathrm{Im}\, k_1 = 0$, hence $\mathrm{Im}\, \psi = 0$; this implies that $\mathrm{Im}\, \phi = H^n_c(W)$, therefore that

$$\mathrm{Im}\, j_1 = \mathrm{Im}\, (j_1 \circ \phi) = \mathrm{Im}\, (\phi' \circ k_2),$$

and our assertion follows from the fact (6.3) that $\mathrm{Im}\, k_2$ is finitely generated.

(ii) Let us now assume that $p^n(x)$ is at most increasingly infinite for all n and x. We have to prove that X is clc.

The space X is n-clc and has property $(P, Q)_n$ for $n > \dim X$. Using induction and (6.2), we may assume that X is $(n+1)$-clc and has property $(P, Q)_j$ for all $j \geq n$.

Let P be an open, relatively compact neighborhood of x, let Q be an open neighborhood of x with closure in P such that $H^n_c(Q \subset P)$ is finitely generated, and let U, V be open neighborhoods of x such that $\overline{V} \subset U$, $\overline{U} \subset Q$. Let $R = Q - \overline{V}$, and let W be an open neighborhood of $\overline{U} - U$ with closure in R. By (5.2), it is enough to show that $H^n(V \subset P)$ is finitely generated.

Let $\alpha: H^n_c(P) \to H^n(P)$ be induced by the inclusion of $C^*_c(P)$ in $C^*(P)$, and let $\beta = \alpha \circ j^*_{QP}$: it follows from the property $(P, Q)_n$ that $\mathrm{Im}\, \beta$ is finitely generated.

Let r_1, r_2 be a partition of unity of $C^*(P)$, subordinate to a covering A_1, A_2 of P such that $\overline{A}_1 \subset U$, $\overline{A}_2 \subset W \cup (P - \overline{U})$. We want to define an L-linear map γ of the module $Z^n(P)$ of cocycles in $C^n(P)$ into $H_c^{n+1}(W)$. Let $z \in Z^n(P)$. Then

$$0 = dz = dr_1 z + dr_2 z,$$

whence

$$S(dr_k z) \subset U \cap (W \cup (P - \overline{U})) = W \qquad (k = 1, 2).$$

Since $S(dr_k z)$ is closed in P, and since it is contained in $\overline{W}$, which is compact and interior to P, $S(dr_k z)$ is also closed in X and is compact. Thus $dr_1 z$ is a cocycle of $C_c^{n+1}(W)$, and we define $\gamma(z)$ to be the cohomology class of $dr_1 z$. By property $(P, Q)_{n+1}$, the image of j^*_{WR} is finitely generated, and we can therefore find a finitely generated submodule A of $Z^n(P)$ which has the same image as $Z^n(P)$ under $j^*_{WR} \circ \gamma$. Let A' be the submodule of $H^n(P)$ defined by A, and let i^* be the natural map of $H^n(P)$ in $H^n(V)$. In order to show that the image of i^* is finitely generated, it is sufficient by the above to prove that it is contained in $i^*\beta(H_c^n(Q)) + i^*A'$.

Let $z' \in H^n(P)$, and let z be a representative cocycle. There exists $a \in A$ such that $j^*_{WR} \circ \gamma(z + a) = 0$, that is, such that

$$dr_1 z + dr_1 a = -dr_2 z - dr_2 a = dc \qquad (c \in C_c^n(R)).$$

Thus

$$z + a = v_1 + v_2,$$

where $v_1 = r_1 z + r_1 a - c$ (resp. $v_2 = r_2 z + r_2 a + c$) is a cocycle with carrier in Q (resp. P - V). Let v_1', v_2' be the corresponding elements of $H^n(P)$. Then $v_1 \in \beta(H_c^n(Q))$, and since v_2' contains a cocycle whose carrier does not meet V, we get

$$i^*(z') + i^*(a') = i^*(v_1'),$$

and therefore

$$i^*(z') \subset i^*(A') + i^*\beta(H_c^n(Q)).$$

We shall come back to this proof in Section 8.

7. GENERALIZED MANIFOLDS

7.1. DEFINITION. *Let* L *be a field,* n *a nonnegative integer. A generalized* n-*manifold* (*an* n-gm) *over* L *is a locally compact, finite-dimensional space* X *in which* $p^n(x, L) = 1$ *and* $p^i(x, L) = 0$ $(i \neq n)$ *at all points.*

This definition is Wilder's [9], except that we do not require the dimension of the space to be equal to n; this mild extension is useful in connection with the Smith theory. Such a space, which we shall call a Wilder n-manifold, is automatically clc by (6.4). In view of (4.2), a paracompact Wilder n-manifold over L satisfies condition (L-n), and we can apply to it the results of Section 3. We now want to relate (3.1) to the duality theorem of [3, 9]. Its formulation in the noncompact case makes use of new homology groups $h_r(X, L)$ and $h^r(X, L)$. In our language, the former is the inductive limit of the cohomology groups $H_c^r(U, L)$ of the open relatively compact subsets of X, with respect to the maps j^*_{UV}, and it is clearly equal to $H_c^r(X, L)$. The space $h^r(X, L)$ is the projective limit of the Čech homology groups $H_r(K, L)$ of the compact subsets of X, with respect to the inclusion maps. The duality theorem then reads (see [3], [9], Chap. VIII, 5.16):

7.2. THEOREM. *Let* X *be a paracompact connected orientable* n-gm. *Then* $h_r(X, L) \cong h^{n-r}(X, L$ $(r \geq 0)$. *Moreover, these spaces have at most countable dimension.*

In the proof, we use the following elementary facts about direct and inverse limits: let

$$V = \lim_{\leftarrow}(V_i, f_{ij}), \qquad W = \lim_{\rightarrow}(W_i, g_{ij}) \qquad (i = 1, 2, \cdots)$$

be respectively inverse and direct limits of sequences of vector spaces over L. Assume that there is a pairing ϕ_i of V_i and W_i and that $f_{ij}: V_j \to V_i$ and $g_{ij}: W_i \to W_j$ are the transposes of each other with respect to ϕ_i, ϕ_j $(1 \leq i \leq j)$. These pairings induce in the obvious way pairings ϕ_{ij} of Im f_{ij} and Im g_{ij} and a pairing ϕ of V and W. Then if ϕ_i and ϕ_j are orthogonal, so is ϕ_{ij}. If all the ϕ_i are orthogonal and if the spaces Im f_{ij} (or equivalently the spaces Im g_{ij}) are finite-dimensional, the pairing ϕ is orthogonal.

Being connected, X is paracompact if and only if it is the union of an increasing sequence of compact subspaces K_i $(i = 1, 2, \cdots)$, and we may assume that K_i is in the interior Int K_{i+1} of K_{i+1}, and that Int K_i is connected $(i = 1, 2, \cdots)$.

Since $H_c^r(\text{Int } K_i \subset \text{Int } K_{i+1}, L)$ is finite-dimensional by (6.2) and (6.4), $h_r(X, L)$ has at most countable dimension. Analogously, it follows from (6.3), or more precisely from its homological counterpart [9, Chap. VI, 7.9], that $h^r(X, L)$ has at most countable dimension.

Let us now define $h_*^r(X, L)$ as the projective limit of the cohomology spaces $H^r(K, L)$ of the compact subspaces of X, with respect to the maps i*. We have here

$$h_*^r(X, L) = \lim_{\leftarrow} H^r(K_i, L) = \lim_{\leftarrow} H^r(\text{Int } K_i, L)\,;$$

and since $H^r(K_i \subset K_{i+1}, L)$ is finite-dimensional by (6.3), $h_*^r(X, L)$ can be considered as a projective limit of finite-dimensional vector spaces and hence has at most countable dimension. Now $H^r(K_i, L)$ and $H_r(K_i, L)$ are paired in the standard way; the pairing commutes with the injection maps, and by our initial remark, it is orthogonal. Hence, by the same remark and the above, we have

$$h_*^r(X, L) \cong h^r(X, L)\,,$$

and (7.2) will follow if we show the existence of an orthogonal pairing of $h_*^{n-r}(X, L)$ with $H_c^r(X, L)$. At least in the case where the Int K_i are paracompact, this follows by the previous argument from (3.1), because the latter implies that

$$H^{n-r}(\text{Int } K_i, L) \cong \text{Hom}\,(H_c^r(\text{Int } K_i, L), L)\,.$$

It also follows then that $H^r(X, L)$ and $h_*^r(X, L)$ are isomorphic if one of them is finite-dimensional. It is not so otherwise, because $h_*^r(X, L)$ has countable dimension, whereas $H^r(X, L)$, being dual to an infinite-dimensional space, has an uncountable base.

In order to derive (3.1) from (7.2) in the case of an orientable n-gm, one should know a priori that $H^r(X, L)$ and $h_*^r(X, L)$ are isomorphic if one of them is finite-dimensional. We do not know whether this can be proved directly, perhaps more generally for a locally compact, paracompact and clc space.

7.3. By (4.2) and Section 6, a paracompact n-gm can also be defined as a space which is clc and satisfies the condition (L-n). Both definitions can be formulated for an arbitrary principal ideal ring (by means of (4.3), in one case); but we do not know whether they are equivalent, or even whether one implies the other when L is not a field, because, seeing no reason why $C_c^*(X, L)$ should be projective, we cannot assert that $H_*(C_*(X, L))$ and $H_c^*(X, L)$ are related by the universal coefficient formula. In the same connection, we may ask the following question: if X satisfies condition (Z-n) (where Z is the ring of integers), and is clc over Z, connected and orientable, does one have, as in the case of ordinary manifolds, an exact sequence

$$0 \to \mathrm{Ext}(H_c^{i+1}(X, Z), Z) \to H^{n-i}(X, Z) \to \mathrm{Hom}(H_c^i(X, Z), Z) \to 0\ ?$$

7.4. *Smith manifolds.* In [8], Smith considers a space X (we shall call it a Smith n-manifold) with the following properties:

(i) X is locally compact, finite-dimensional and clc over L.

(ii) Property P_n: Each $x \in X$ has an open neighborhood U such that (a) given $y \in U$ and an open neighborhood V of y in U, there is an open neighborhood W of y in V such that

$$H_c^i(W \subset V, L) = 0 \ (i \neq n), \qquad H_c^n(W \subset V, L) \cong L;$$

(b) for each open V in U, $H_c^n(V \subset U, L) \neq 0$.

(iii) Property Q: Given x and an open U containing x, there exists an open V with $x \in V \subset U$ such that, given $y \in V$ and an open neighborhood U' of y contained in V, there exists an open neighborhood V' of y in U' such that the map of relative cohomology modules $H^*(X - V', X - V, L) \to H^*(X - U', X - U, L)$ induced by the inclusion $(X - U', X - U) \subset (X - V', X - V)$ has zero image.

Actually, Smith considers this only when L is the field of integers modulo a prime p, and X is compact and also locally paracompact; and he formulates his conditions in homology. The extension of it mentioned here has been studied by Yang [11], who expresses it in Čech homology with a compact abelian group of coefficients; but there is of course no difficulty in going over to cohomology. The property (iia) means that $p^n(x, L) = 1$, $p^i(x, L) = 0$ $(i \neq n)$; therefore, by (6.4), the condition clc is redundant, and X is a Wilder n-manifold. In view of (iia), (iib) clearly implies that x has a connected neighborhood U such that $H_c^n(U, L)$ contains a nonzero element which belongs to $H_c^n(V \subset U, L)$ for each open V in U. Thus (iib) implies local orientability (see 3.4). Yang has shown [11, Appendix] that (iii) follows from the other conditions, so that the notions of a Smith n-manifold and of a locally orientable Wilder n-manifold are equivalent. In [11], Corollary 3.3 of the present paper is proved directly, for a Smith n-manifold, for any principal ideal ring L and any open subspace U. As a consequence, if a proper closed subset of a connected Smith n-manifold is a Wilder m-manifold, then $m < n$; in fact, for any boundary point x of F and any connected neighborhood U of x in X, we have $H_c^j(F \cap U, L) = 0$ for $j \geq n$, and therefore the jth local Betti number of F around x is zero for all $j \geq n$.

8. A MAYER-VIETORIS SEQUENCE IN Φ-COHOMOLOGY

The map γ in the proof of (6.4ii) also defines a map γ^* of $H^n(P)$ in $H_c^{n+1}(W)$, leading to an exact sequence of the Mayer-Vietoris type, which we shall now discuss briefly.

Let Φ be a family of closed sets in X, satisfying the usual conditions for Φ-cohomology [4b, Exp. XVII]; and let A be a fundamental grating over L for X, (for example, let $A = C^*(X, L)$), and let A_Φ be the subgrating of elements in A with carriers in Φ. Then the Φ-cohomology group $H^*_\Phi(X, L)$ of X with coefficients in L is by definition $H^*(A_\Phi)$. If Y is an open subspace of X, and Φ' is the set of elements of Φ contained in Y, then $H^*(A_{\Phi'}) = H^*_{\Phi'}(Y, L)$, and the inclusion of $A_{\Phi'}$ in A_Φ defines a homomorphism of $H^*_{\Phi'}(Y, L)$ in $H^*_\Phi(X, L)$; this homomorphism generalizes j^*_{YX}, and it will be denoted by the same symbol.

Let now X be the union of two open subspaces X_1, X_2, and let Φ_1, Φ_2, Φ_{12} be the family of elements in Φ contained in X_1, X_2, $X_1 \cap X_2$, respectively. A map $\gamma^*\colon H^n_\Phi(X, L) \to H^{n+1}_{\Phi_{12}}(X_1 \cap X_2, L)$ is defined in the following way: let $z' \in H^n(X, L)$ and z be a representative cocycle. Consider a decomposition $z = z_1 + z_2$ of z with $S(z_i) \subset \Phi_i$ $(i = 1, 2)$; using a partition of unity as in (6.4ii), one sees that there is always at least one such decomposition. Then $dz_1 + dz_2 = 0$, hence $S(dz_1) \subset X_1 \cap X_2$, and $\gamma^*(z')$ is by definition the class of dz_1 in $H^{n+1}_{\Phi_{12}}(X_1 \cap X_2, L)$. It is a routine exercise to verify that $\gamma^*(z')$ does not change if we use another decomposition of z, or another cocycle of z', and that the following sequence is exact:

$$\to H^n_{\Phi_{12}}(X_1 \cap X_2, L) \xrightarrow{j^*} H^n_{\Phi_1}(X_1, L) + H^n_{\Phi_2}(X_2, L) \xrightarrow{k^*} H^n_\Phi(X, L) \to H^{n+1}_{\Phi_{12}}(X_1 \cap X_2, L) \to ,$$

where

$$j^*(x) = j^*_{U,X_1}(x) - j^*_{U,X_2}(x) \qquad (U = X_1 \cap X_2,\ x \in H^n_{\Phi_{12}}(U, L)),$$

$$k^*(x_1 + x_2) = j^*_{X_1,X}(x_1) + j^*_{X_2,X}(x_2) \qquad (x_i \in H^n_{\Phi_i}(X_i, L),\ i = 1, 2).$$

If $X = Y_1 \cup Y_2$, with Y_i open and containing X_i, one obtains in the obvious way a map of the Mayer-Vietoris sequence of (X, X_1, X_2) into that of (X, Y_1, Y_2). When Φ is the family of compact subspaces, we get the Mayer-Vietoris sequence for cohomology with compact carriers used in (6.2).

We now use the notations of (6.4ii) and take for Φ the family of all closed sets on P. Then the elements of Φ with carriers in Q (resp. W, R) are precisely the compact subsets of Q (resp. W, R) and we have a commutative diagram

$$\begin{tikzcd}
H_c^{n+1}(W) \arrow[rr, "j^*_{WR}"] & & H_c^{n+1}(R) \\
& H^n(P) \arrow[ul, "\gamma_1^*"] \arrow[ur, "\gamma_2^*"'] & \\
& H_c^n(Q) + H_\psi^n(P - \overline{V}) \arrow[u, "k^*"] &
\end{tikzcd} ,$$

where γ_1^* is the map of the Mayer-Vietoris sequence of $(P, U, W \cup (P - \overline{U}))$, and Ψ is the family of elements of Φ contained in $P - \overline{V}$. Then (6.4ii) follows from the facts that $\operatorname{Im} j^*_{WR}$ and $k^*(H^n_c(Q)) = \beta(H^n_c(Q))$ are finitely generated and that $i^*k^*(H^n_\Psi(P - \overline{V})) = 0$.

In this proof, we assume tacitly that P is paracompact, since the elements of a family Φ are paracompact; but in fact this condition does not play any role here if we take $A = C^*(P, L)$; in any case, the first version of the proof did not require that assumption.

REFERENCES

1. P. Alexandroff, *On local properties of closed sets*, Ann. of Math. (2) 36 (1935), 1-35.
2. E. G. Begle, *Locally connected spaces and generalized manifolds*, Amer. J. Math. 64 (1942), 553-574.
3. ———, *Duality theorems for generalized manifolds*, Amer. J. Math. 67 (1945), 59-70.
4. H. Cartan, *Séminaire de topologie algébrique de l'Ecole Normale Supérieure*, a) 1948-49, b) 1950-51.
5. H. Cohen, *A cohomological definition of dimension for locally compact Hausdorff spaces*, Duke Math. J. 21 (1954), 209-224.
6. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton Mathematical Series 15 (1952).
7. E. E. Floyd, *Closed coverings in Čech homology theory*, Trans. Amer. Math. Soc. 84 (1957), 319-337.
8. P. A. Smith, *Transformations of finite period. II*, Ann. of Math. (2) 40 (1939), 690-711.
9. R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications 32 (1949).
10. ———, *Some consequences of a method of proof of J. H. C. Whitehead*, Michigan Math. J. 4 (1957), 27-31.
11. C. T. Yang, *Transformation groups on a homological manifold*, Trans. Amer. Math. Soc. (to appear).

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