SOME REMARKS ON ELEMENTARY DIVISOR RINGS II

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1. INTRODUCTION

A commutative ring S with identity element 1 is called an *elementary divisor ring* (resp. *Hermite ring*) if for every matrix A over S there exist nonsingular matrices P, Q such that PAQ (resp. AQ) is a diagonal matrix (resp. triangular matrix). It is clear that every elementary divisor ring is an Hermite ring, and that every Hermite ring is an F-ring (that is, a commutative ring with identity in which all finitely generated ideals are principal).

We are concerned, in this paper, with identifying those F-rings that are elementary divisor rings. It is known that every F-ring satisfying the ascending chain conditions on ideals is an elementary divisor ring [5, Theorem 12.3, ff]. The earliest affirmative result of this kind obtained without chain conditions is Helmer's result that every *adequate ring* without (proper) divisors of 0 is an elementary divisor ring [3]. (An F-ring S is an *adequate ring* if, for every $a, b \in S$ with $a \neq 0$, we may write $a = rs$ with $(r, b) = (1)$ and with $(t, b) \neq (1)$ for every nonunit divisor t of s. As usual, $(a_1, \cdots, a_n)$ denotes the ideal generated by $a_1, \cdots, a_n$.) This result was generalized successively by Kaplansky [5, Theorem 5.3] and by L. Gillman and the author [1]. The latter showed that every adequate Hermite ring is an elementary divisor ring, and they gave examples [2, Corollary 6.7] of elementary divisor rings that are not adequate rings. In addition they gave examples of F-rings that are not Hermite rings, and of Hermite rings that are not elementary divisor rings [2, Examples 3.4 and 4.11]. All of these examples have divisors of 0. We give below what seems to be the first known example of an elementary divisor ring without divisors of 0 that is not an adequate ring.

In addition, by using theorems in [1] and [5], we obtain the following affirmative results: 1. If the Perlis-Jacobson radical [4] R(S) of the F-ring S contains a prime ideal of S, then S is an Hermite ring. 2. In order that an Hermite ring S be an elementary divisor ring, it is enough that S/R(S) be an elementary divisor ring. 3. Every nonzero (proper) prime ideal of an adequate ring S is contained in a unique maximal ideal of S. 4. If S is an Hermite ring and every element of S not in R(S) is contained in at most a finite number of maximal ideals, then S is an elementary divisor ring.

In the last section of the paper, we give the example cited above and state some unsolved problems.

2. THE AFFIRMATIVE RESULTS

The following theorem, which is proved in [1], is used repeatedly below.

THEOREM 1. (a) *A commutative ring with identity is an Hermite ring if and only if it satisfies the condition* (T): *for all* $a, b \in S$, *there exist* $a_1, b_1, d \in S$ *such that* $a = a_1 d$, $b = b_1 d$, *and* $(a_1, b_1) = (1)$.

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(b) *A commutative ring* S *with identity is an elementary divisor ring if and only if it is an Hermite ring that satisfies the condition* (D'): *for all* a, b, c $\in$ S *with* (a, b, c) = (1), *there exist* p, q $\in$ S *such that* (pa, pb + qc) = (1).

In what follows, R(S) will denote the Perlis-Jacobson radical of S [4]. Since S will always denote a commutative ring with identity, R(S) is the intersection of all the maximal ideals of S. We observe at this point that if $r \in R(S)$, then $1 + r$ is a unit. (For if $1 + r$ were in a maximal ideal M of S, then 1 would be in M.)

THEOREM 2. *If the Perlis-Jacobson radical* R(S) *of the* F*-ring* S *contains a prime ideal of* S, *then* S *is an Hermite ring.*

Proof. We will show that condition (T) of Theorem 1(a) holds for any pair a, b of elements of S. Two cases will be considered.

Case (i). At least one of a, b is not in R(S). Then, since S is an F-ring, there is a $d \in S$ such that (a, b) = (d) and $d \notin R(S)$. Since (a, b) = (d), there exist $a_1, b_1, s, t \in S$ such that $a = a_1d$, $b = b_1d$, and $sa + tb = d$. Hence $(sa_1 + tb_1 - 1)d = 0$. But R(S) contains a prime ideal of S, and $d \notin R(S)$, so $(sa_1 + tb_1 - 1) \in R(S)$, whence $sa_1 + tb_1$ is a unit. Thus, $(a_1, b_1) = (1)$.

Case (ii). Both a and b are in R(S). The case a = b = 0 yields no difficulty, so we assume that at least one of a, b is nonzero. As in case (i), there exist a_1, b_1, s, t, d in S such that $a = a_1d$, $b = b_1d$, and $sa + tb = d$. If both a_1 and b_1 are in R(S), then $(sa_1 + tb_1 - 1)$ is a unit, whence from $(sa_1 + tb_1 - 1)d = 0$ we obtain d = 0, contrary to assumption. Hence at least one of a_1, b_1 is not in R(S). By case (i), there exist $a_1', b_1', d' \in S$ such that $a_1 = a_1'd'$, $b_1' = b_1'd'$, and $(a_1', b_1') = (1)$. Then $a = a_1'(d'd)$, $b = b_1'(d'd)$ and $(a_1', b_1') = (1)$.

Thus, in both cases, condition (T) holds for any $a, b \in S$. Hence S is an Hermite ring.

Theorem 2 should be compared with Kaplansky's result [5, Theorem 3.2] that every F-ring S whose divisors of 0 are in R(S) is an Hermite ring.

THEOREM 3. *An Hermite ring* S *is an elementary divisor ring if and only if* S/R(S) *is an elementary divisor ring.*

Proof. Obviously every homomorphic image of an elementary divisor ring is an elementary divisor ring, so we need only prove the sufficiency. In particular, it is only necessary to show that condition (D') of Theorem 1(b) holds in S if it holds in S/R(S).

Suppose $a, b, c \in S$ are such that (a, b, c) = (1) (in S). By hypothesis and Theorem 1(b), there exist $p, q, u, v \in S$ and $r \in R(S)$ such that $u(pa) + v(pb + qc) = 1 + r$. But $1 + r$ is a unit of S, so (pa, pb + qc) = (1) (in S). Hence, by Theorem 1(b), S is an elementary divisor ring.

COROLLARY 1. *If* S *is an* F*-ring whose Perlis-Jacobson radical* R(S) *contains a prime ideal of* S, *then* S *is an elementary divisor ring if and only if* S/R(S) *is an elementary divisor ring.*

Proof. This is an immediate consequence of Theorems 2 and 3.

THEOREM 4. *Every nonzero (proper) prime ideal of an adequate ring* S *is contained in a unique maximal ideal of* S.

Proof. Suppose the nonzero prime ideal P of the F-ring S is contained in the intersection of two distinct maximal ideals M_1, M_2 of S. Since M_1, M_2 are distinct maximal ideals, there exist $m_1 \in M_1$, $m_2 \in M_2$ such that $(m_1, m_2) = (1)$. Let p be

any nonzero element of P. If $p = rs$ with $(r, m_1) = (1)$, then, since P is a prime ideal and $P \subset M_1$, it follows that $s \in P$. Since S is an F-ring, there is a $d \in S$ such that $(d) = (s, m_2)$. Since $P \subset M_2$, d is a nonunit divisor of s. But $(d, m_1) \supset (m_2, m_1) = (1)$, so S is not an adequate ring.

We next generalize the theorem in [1] that every adequate Hermite ring is an elementary divisor ring. The proof is patterned after that of [5, Theorem 5.3].

Definition. If S is a commutative ring with identity, and $a \in S$, let Z(a) denote the set of maximal ideals of S that contain a.

THEOREM 5. *If* S *is an Hermite ring such that, for every* a, $c \in S$ *with* $a \notin R(S)$, *there is an* $r \in S$ *such that* $Z(r) = \{Z(a) \sim Z(c)\}$, *then* S *is an elementary divisor ring.*

Proof. Since the conditions given in the hypothesis are preserved under homomorphism, by Theorem 3, we may assume that S is semisimple. We will verify that condition (D') of Theorem 1(b) holds in S. Assume that a, b, $c \in S$ and $(a, b, c) = (1)$. The case $a = 0$ yields no difficulty, so we assume that $a \neq 0$. By hypothesis, there is an $r \in S$ such that $Z(r) = \{Z(a) \sim Z(c)\}$. Then $(c, r) = (1)$, so there exists a $q \in S$ such that $(b + qc, r) = (1)$. Since S is an F-ring, there is a $d \in S$ such that $(d) = (a, b + qc)$. If d is not a unit, then there is a maximal ideal M of S such that $M \in Z(d)$. Then $M \in Z(a)$ and $M \in Z(b + qc)$. Hence $M \notin Z(r) = \{Z(a) \sim Z(c)\}$, so $M \in Z(c)$. But then $b \in M$, contrary to the assumption that $(a, b, c) = (1)$. Thus, S is an elementary divisor ring.

COROLLARY 2. *If* S *is an* F*-ring with infinitely many distinct maximal ideals, or an Hermite ring with only a finite number of distinct maximal ideals, and if for every* $a \in S \sim R(S)$ *the set* Z(a) *is finite, then* S *is an elementary divisor ring.*

Proof. We show first that, in case S has infinitely many distinct maximal ideals, S is an Hermite ring. If $a, b \in S \sim R(S)$, then $Z(ab) = Z(a) \cup Z(b)$ is a finite set, so $ab \notin R(S)$. Hence R(S) is a prime ideal of S, so, by Theorem 2, S is an Hermite ring.

We next verify the hypothesis of Theorem 5. By Theorem 3, there is no loss of generality in assuming that S is semisimple. It will be shown first that for every maximal ideal M of S, there is an $m \in S$ such that $Z(m) = \{M\}$. Choose $a \neq 0$ in S such that $a \in M$. Suppose that $(Z(a) \sim \{M\}) = \{M_1, \cdots, M_n\}$. For each $i = 1, 2, \cdots, n$, there is an $e_i \in (M \sim M_i)$. Since S is an F-ring, there is an $m \in S$ such that $(m) = (a, e_1, \cdots, e_n)$. Clearly, $Z(m) = \{M\}$.

If $a, c \in S$ with $a \neq 0$, then by hypothesis $\{Z(a) \sim Z(c)\}$ is a finite set. The case when this set is empty is trivial, so we assume that $(Z(a) \sim Z(c)) = \{M_1, \cdots, M_n\}$ is nonempty. By the above, for each $i = 1, 2, \cdots, n$, then is an $m_i \in M_i$ such that $Z(m_i) = \{M_i\}$. Then, if $r = m_1 m_2 \cdots m_n$, then $Z(r) = (Z(a) \sim Z(c))$. Thus, by Theorem 5, S is an elementary divisor ring.

3. EXAMPLES AND PROBLEMS

We begin this section with an example of an integral domain that is an elementary divisor ring, but is not an adequate ring.

EXAMPLE 1. Let S denote any elementary divisor ring, that is an integral domain with more than one maximal ideal, and let N denote its field of formal quotients. (For example, S could be the ring of (rational) integers, N the field of

rational numbers.) Let P denote the ring of formal power series over N in an indeterminate x. Each element $a \in P$ has a unique representation of the form $a = \sum_{k=0}^{\infty} a_k x^k$, where $a_k \in N$. Let $Q = \{a \in P: a_0 \in S\}$. Clearly Q is a sub-ring of the integral domain P. It will be proved that Q is an elementary divisor ring that is not an adequate ring.

First we show that Q is an F-ring by verifying that, for each $a, b \in Q$, the ideal (a, b) is principal. The case when a or b is 0 is trivial, so we assume that neither a nor b is 0. For any nonzero $c \in Q$, let n(c) denote the least (nonnegative) integer such that $c_{n(c)} \neq 0$. If $c^* = c_{n(c)} x^{n(c)}$, then

$$c = c^* \left(1 + \sum_{k=n(c)+1}^{\infty} \frac{c_k}{c_{n(c)}} x^k \right).$$

Since the last factor is a unit of Q, (c) = (c*). By the above, (a, b) = (a*, b*). If $n(a) > n(b)$, b* is a divisor of a*, so (a*, b*) = (b*). If n(a) = n(b) = n, write

$$a^* = \frac{\alpha_1}{\alpha_2} x^n, \quad b^* = \frac{\beta_1}{\beta_2} x^n,$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in S$, and $\alpha_2 \beta_2 \neq 0$. (If n = 0, take $\alpha_2 = \beta_2 = 1$.) Since S is an Hermite ring, there exist (Theorem 1(a)) $\gamma_1, \gamma_2, \delta \in S$ such that

$$\alpha_1 \beta_2 = \gamma_1 \delta, \qquad \alpha_2 \beta_1 = \gamma_2 \delta, \qquad (\gamma_1, \gamma_2) = (1).$$

It is easily verified that $(a^*, b^*) = \left(\frac{\delta x^n}{\alpha_2 \beta_2} \right)$. This completes the proof that Q is an F-ring.

It is easy to verify that $R(Q) = \{a \in Q: a_0 = 0.\}$. Since R(Q) is a prime ideal, and R/R(Q) and S are isomorphic, it follows from Corollary 1 that Q is an elementary divisor ring.

The subring $S' = \{a \in Q: a = a_0\}$ of Q is isomorphic with S, and every maximal ideal of Q consists of the ideal of Q generated by a maximal ideal of S'. Hence, since R(Q) is a prime ideal of Q, and S contains more than one maximal ideal, it follows from Theorem 4 that Q is not an adequate ring.

EXAMPLE 2. An elementary divisor ring that does not satisfy the hypothesis of Theorem 5. It is easily seen that the elementary divisor ring of [2, Example 4.9] will do.

We close the paper with the statement of several problems, the first of which seems to be difficult.

1. Is every F-ring that is an integral domain an elementary divisor ring?

2. If S is an F-ring, and S/R(S) is an Hermite ring, need S be an Hermite ring? (An affirmative answer to this question would imply, in view of Theorem 3, that S is an elementary divisor ring if and only if S/R(S) is an elementary divisor ring.)

3. If S is an F-ring with only a finite number of distinct maximal ideals, need S be an Hermite ring—hence, by Corollary 2, an elementary divisor ring?

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