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SOME SYSTEMS OF NATURAL DEDUCTION

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Gentzen type systems of natural deduction have received much attention because they have some nice structural properties, such as the subformula property. In this paper, however, the primary concern will be with naturalness and economy.

1. *Propositional Calculi.* Let $\sim$ (negation) and $\supset$ (implication) be the only primitive logical symbols in a propositional calculus whose primitive rules of inference are defined inductively as follows:

(1) Premise Introduction (**PI**): $\{A\} \vdash A$.
(2) Premise Engagement (**PE**): If $\Gamma \vdash A$, then $\Gamma - \{B\} \vdash B \supset A$.
(3) Modus Ponens (**MP**): If $\Gamma \vdash A$ and $\Delta \vdash A \supset B$, then $\Gamma \cup \Delta \vdash B$.
(4) Negation Ponens (**NP**): If $\Gamma \vdash A$ and $\Delta \vdash B \supset \sim A$, then $\Gamma \cup \Delta \vdash B$.

Let us consider the following alternate rules:

(1.1) Strong Premise Introduction (**SPI**): $\{A\} \vdash B \supset A$.
(2.1) Actual Premise Engagement (**APE**): If $\Gamma \cup \{B\} \vdash A$, then $\Gamma \vdash B \supset A$.
(2.2) Weak Premise Engagement (**WPE**): If $\Gamma \cup \{B\} \vdash A$, then $\Gamma - \{B\} \vdash B \supset A$.

We easily obtain the following.

Theorem 1. *The systems* {**PI,PE,MP,NP**} *and* {**SPI,WPE,MP,NP**} *are classical, i.e., they are complete.*

Proof: Cf. [1] pp. 119.

Observe that {**PI,PE,MP**} yields **APE**, and **APE** yields **WPE**. **APE** corresponds to the deduction theorem. Also,

Theorem 2. {**PI,WPE,MP,NP**} *is not classical.*

Proof: Consider the equivalent system {**PI,WPE,MP,NP***} where **NP*** states: If $\Gamma \vdash A$ and $\Delta \cup \{\sim B\} \vdash \sim A$, then $\Gamma \cup \Delta \vdash B$. This latter system obeys the subformula property (hint: consider a maximal formula in the proof which is not a subformula of the last step and eliminate it. Cf. Dag Prawitz [3]). Hence $A, B \vdash A$ cannot be proved.

Let v (disjunction) be primitive instead of $\supset$, define $A \supset B$ as $\sim A \vee B$, and consider the rules:

(4.1) Disjunction supplement (**DS**): $A \supset B \vdash A \vee C \supset C \vee B$.

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(4.2) Left Disjunct (**LD**): $A \vdash A \vee B$.
(4.3) Right Disjunct (**RD**): $B \vdash A \vee B$.
(4.4) Proof by Cases (**PC**): $A \supset C$, $B \supset C$, $A \vee B \vdash C$[1].

Theorem 3. {**PI,PE,MP,DS**} *and* {**PI,WPE,LD,RD,PC**} *are classical*[2].

Proof: The latter system easily yields {**PE,MP,DS**}, so let us consider the former. Rule **DS** immediately gives $A \vee C \vdash {\sim}A \vee B \supset C \vee B$. Replacing A by ${\sim}A$, C by A, and B by ${\sim}B$, and observing that $\vdash {\sim}A \vee A$, we obtain $\vdash {\sim}A \supset {\sim}B . \supset . A \vee {\sim}B$. Now, in rule **DS**, replace B by A and C by ${\sim}B$ to obtain $\vdash A \vee {\sim}B \supset {\sim}B \vee A$. Hence we obtain $\vdash {\sim}A \supset {\sim}B . \supset . B \supset A$ so rule **NP** is available. Write $A \vee^* B$ for ${\sim}A \supset B$ (i.e. for ${\sim}{\sim}A \vee B$). Then, by Theorem 1, every tautology that can be written in terms of $\sim$ and $\vee^*$ is provable in {**PI,PE,MP,DS**}. For every formula A in {**PI,PE,MP,DS**}, let A^* be the formula obtained from A by replacing each occurrence of $\vee$ in A by $\vee^*$. If A is a tautology, then A^* is a tautology. By induction on the number of logical connectives in A, we can show that $\vdash A^* \supset A$ in {**PI,PE,MP,DS**}. Hence if A is a tautology, then $\vdash A$ in {**PI,PE,MP,DS**}. Q.E.D.

Now let & (conjunction) be primitive instead of $\supset$ or $\vee$, define $A \supset B$ as ${\sim}(A \,\&\, {\sim}B)$, and consider the rules:

(4.5) Conjunction Supplement (**CS**): $A \supset B \vdash A \,\&\, C \supset C \,\&\, B$.
(4.6) Rosser's Rule (**RR**): $A \supset B \vdash {\sim}(B \,\&\, C) \supset {\sim}(C \,\&\, A)$[3].
(4.7) Left Conjunct (**LC**): $A \,\&\, B \vdash A$.
(4.8) Right Conjunct (**RC**): $A \,\&\, B \vdash B$.
(4.9) Conjunction Formation (**CF**): A, $B \vdash A \,\&\, B$.
(4.10) Conjunction Commutation (**CC**): ${\sim}(A \,\&\, B) \vdash {\sim}(B \,\&\, A)$[4].
(4.11) Left Ponens (**LP**): A, ${\sim}({\sim}B \,\&\, A) \vdash B$.

We have

Theorem 4. {**PI,PE,MP,NP,CS**} *and* {**PI,PE,MP,RR**} *are classical*[2].

Proof: Rule **RR** of {**PI,PE,MP,RR**} immediately gives ${\sim}(B \,\&\, C) \vdash {\sim}(A \,\&\, {\sim}B) \supset {\sim}(C \,\&\, A)$. Replacing A by ${\sim}A$, B by ${\sim}B$, and C by B gives ${\sim}({\sim}B \,\&\, B) \vdash {\sim}({\sim}A \,\&\, {\sim}{\sim}B) \supset {\sim}(B \,\&\, {\sim}A)$. Replacing A by B and C by ${\sim}B$ in rule **RR** gives $B \supset B \vdash {\sim}(B \,\&\, {\sim}B) \supset {\sim}({\sim}B \,\&\, B)$. Hence we obtain $B \supset B \vdash {\sim}A \supset {\sim}B . \supset . B \supset A$. But $\vdash B \supset B$, hence rule **NP** is available in {**PI,PE,MP,RR**}. Also, we obtain $\vdash {\sim}(B \,\&\, C) \supset {\sim}(C \,\&\, A) . \supset . C \,\&\, A \supset B \,\&\, C$, so $A \supset B \vdash C \,\&\, A \supset B \,\&\, C$ in {**PI,PE,MP,RR**}. Replacing B by A gives $\vdash C \,\&\, A \supset A \,\&\, C$. So rule **CS** is available in {**PI,PE,MP,RR**}, hence {**PI,PE,MP,RR**} yields {**PI,PE,MP,NP,CS**}. Now we proceed as in Theorem 3 to show that {**PI,PE,MP,NP,CS**} is classical, writing $A \,\&^*\, B$ for ${\sim}(A \supset {\sim}B)$ (i.e. for ${\sim}{\sim}(A \,\&\, {\sim}{\sim}B)$).

Theorem 5. {**PI,WPE,MP,LC,RC,CF,LP**} *and* {**PI,WPE,MP,LC,CF,CC**} *are classical.*

Proof: We shall only show that the latter system is classical, the former system will then follow easily. We use the device of numbering premises as appears in [7].

*1.	${\sim}({\sim}A \,\&\, B)$, $B \vdash A$.		**LP**
	{1}	(1) ${\sim}({\sim}A \,\&\, B)$	**PI**

{2}	(2)	B	PI
{1}	(3)	$\sim(B\&\sim A)$	CC, 1
{1,2}	(4)	A	MP, 2, 3

*2. *If A and B are distinct, then* $A \vdash B \supset A$.

{1}	(1)	A	PI
{2}	(2)	B	PI
{1,2}	(3)	$A\&B$	CF, 1, 2
{1,2}	(4)	A	LC, 3
{1}	(5)	$B\supset A$	WPE, 4

*3. $\sim\sim A \vdash A$.

{1}	(1)	$\sim\sim A$	PI
{2}	(2)	$\sim A$	PI
	(3)	$\sim(\sim A\&\sim\sim A)$	WPE, 2
{1}	(4)	A	LP, 1, 3

*4. $A \vdash \sim(\sim\sim\sim A\&\sim\sim\sim A)$.

{1}	(1)	A	PI
{1}	(2)	$\sim(\sim\sim\sim A\&\sim A)$	*2, 1
{3}	(3)	$\sim\sim\sim A$	PI
{3}	(4)	$\sim A$	*3, 3
{1,3}	(5)	$\sim\sim A$	LP, 2, 4
{1}	(6)	$\sim(\sim\sim\sim A\&\sim\sim\sim A)$	WPE, 5

*5. $A \vdash \sim\sim A$.

{1}	(1)	A	PI
{2}	(2)	$\sim\sim\sim A$	PI
{2}	(3)	$\sim\sim\sim A\&\sim\sim\sim A$	CF, 2
	(4)	$\sim(\sim\sim\sim A\&\sim(\sim\sim\sim A\&\sim\sim\sim A))$	WPE, 3
{1}	(5)	$\sim(\sim\sim\sim A\&\sim\sim\sim A)$	*4, 1
{1}	(6)	$\sim\sim A$	LP, 4, 5

*6. $\sim A \vdash \sim(A\&B)$.

{1}	(1)	$\sim\sim(A\&B)$	PI
{1}	(2)	$A\&B$	*3, 1
{1}	(3)	A	LC, 2
	(4)	$\sim(\sim\sim(A\&B)\&\sim A)$	WPE, 3
{5}	(5)	$\sim A$	PI
{5}	(6)	$\sim(A\&B)$	LP, 4, 5

*7. $\sim B \vdash \sim(A\&B)$.

*8. $\sim(\sim A\&\sim A) \vdash A$.

{1}	(1)	$\sim(\sim A\&\sim A)$	PI
{2}	(2)	$\sim A$	PI
{2}	(3)	$\sim A\&\sim A$	CF, 2
	(4)	$\sim(\sim A\&\sim(\sim A\&\sim A))$	WPE, 3
{1}	(5)	A	LP, 1, 4

*9. $A \supset B,\ {\sim}A \supset B \vdash B.$

$\{1\}$	(1)	${\sim}(A \& {\sim}B)$	PI
$\{2\}$	(2)	${\sim}({\sim}A \& {\sim}B)$	PI
$\{3\}$	(3)	${\sim}B$	PI
$\{2,3\}$	(4)	A	LP, 2, 3
$\{1,2,3\}$	(5)	B	MP, 1, 4
$\{1,2\}$	(6)	${\sim}({\sim}B \& {\sim}B)$	WPE, 5
$\{1,2\}$	(7)	B	*8, 6

Using *5, *6, *7, **CF**, and *9 we can prove that if A is a tautology built up from $\sim$ and $\&$, then ${\sim}A$ in {**PI**,**WPE**,**MP**,**LC**,**CF**,**CC**}. Q.E.D.

Theorem 6. {**PI**,**PE**,**MP**,**LC**,**RC**,**CF**,**CS**} *is not classical*[5].

Proof: Consider the truth table

&	*1*	*2*	*3*	$\sim$
1	*1*	*2*	*3*	*3*
2	*1*	*2*	*3*	*2*
3	*3*	*3*	*3*	*1*

where 1 and 2 are the designated values.

2. *Predicate Calculi.* If the universal quantifier is the only added primitive logical operator to, say, $\sim$ and $\supset$, then consider the rules:

(5) Universal Generalization (**UG**): *If* $\Gamma \vdash A$, *then* $\Gamma \vdash (x)A$ *provided x does not occur free in* Γ[6].

(6) Universal Specialization (**US**): $(x)A \vdash \mathbf{S}_t^x A|$ *provided t is a term free for x in A*[7].

We easily have

Theorem 7. {**PI**,**PE**,**MP**,**NP**,**UG**,**US**} *is classical.*

Now let the existential quantifier be primitive instead of the universal and consider the rules:

(5.1) Existential Generalization (**EG**): $\mathbf{S}_t^x A| \vdash (Ex)A$ *provided t is a term free for x in A.*

(6.1) Existential Specialization (**ES**): $(Ex)A \vdash \mathbf{S}_y^x A|$ *provided* $\mathbf{S}_x^y \mathbf{S}_y^x A|$ *is A and, if $\Gamma \vdash B$ is a previous step of the proof which also results from rule* **ES**, *then y does not occur free in B (y is called a restricted variable of the proof)*[8].

We have

Theorem 8. {**PI**,**PE**,**MP**,**NP**,**EG**,**ES**} *is classical, where we agree that a particular proof is not complete unless no restricted variable of the proof occurs free in its last step*[9].

Quine, in [4] and [5], has introduced alternate rules **UG** and **ES**, modifications of which we shall now consider:

(5.2) Quine's Universal Generalization (**QUG**): *If* $\Gamma \vdash \underset{\bullet}{\mathsf{S}}^{x}_{y} A|$, *then* $\Gamma \vdash (x)A$ *provided* $\underset{\bullet}{\mathsf{S}}^{y}_{x}\, \underset{\bullet}{\mathsf{S}}^{x}_{y} A|$ *is* A (*y is called a flagged variable of the proof*), *y has not been* f l a g g e d *previously in the proof, and* (a) *If* $\Gamma_1 \vdash A_1, \ldots, \Gamma_n \vdash A_n$ *are previous steps of the proof which also result from rule* **QUG** *where* $y_1, \ldots, y_n$ *are the flagged variables respectively, then either y does not occur free in* A_1, *or* y_1 *does not occur free in* $A_2, \ldots,$ *or* y_{n-1} *does not occur free in* A_n, *or* y_n *does not occur free in* A.

(6.2) Quine's Existential Specialization (**QES**): $(Ex)A \vdash \underset{\bullet}{\mathsf{S}}^{x}_{y} A|$ *provided* $\underset{\bullet}{\mathsf{S}}^{y}_{x}\, \underset{\bullet}{\mathsf{S}}^{x}_{y} A|$ *is* A *and* (b) (*where* (b) *results from* (a) *by replacing* "**QUG**" *by* "**QES**").

Theorem 9. *If the only quantifier primitive is the universal, then* {**PI,PE,MP,NP,QUG,US**} *is classical; if only the existential, then* {**PI,PE, MP,NP,EG,QES**} *is classical. In either case, we agree that a particular proof is not complete unless no flagged variable of the proof occurs free in its last step.*

NOTES

1. These rules have been defined in an abbreviated manner, which is possible in any system in which { **PI, WPE, MP** } holds.
2. This theorem is due to Rosser, private communication late 1960 or early 1961.
3. This is essentially axiom scheme 3 of [4] p. 55.
4. This rule was suggested by Rosser.
5. This answers a question raised by Rosser.
6. This is essentially rule G of [6] p. 124.
7. See [1] p. 170 and [2] p. 79. We need make no restrictions on the richness of our object language, in particular it may be composed of various types and various sorts and even contain variable-binding operators such as λ, ι, etc. Of course, in such cases, our definition of $\underset{\bullet}{\mathsf{S}}$ would have to be extended.
8. This is essentially rule C of [6] pp. 129.
9. Of course, "unless" is a connective for exclusive disjunction.

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