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NOTE TO MY PAPER: "A DIAGRAM OF THE FUNCTORS OF THE TWO-VALUED PROPOSITIONAL CALCULUS"

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Theorem 6 of my paper [1] gives a criterion for determining which functors of the two-valued propositional calculus are Sheffer functors, viz. that a functor lie in the N quadrant and not be on the vertical axis. From the nature of the diagram presented, it is obvious that precisely one fourth of the n-ary functors lie in the N quadrant, which means that there are 2^{2^n-2} of them. (Note that, since the I, U, O, and N quadrants in the diagram correspond respectively to the α, β, γ, and δ functors of Post [2], we have a simple formula for the number of such functors)

On the other hand, we can see that the total number of n-ary functors lying on the vertical axis is the square of the number of $(n-1)$-ary functors on the vertical axis, which yields almost immediately that there are $2^{2^{n-1}}$ of them. Exactly half of these are in the N quadrant, which gives us this equation for the number of n-ary functors satisfying the criterion:

$$\mathbf{S}(n) = 2^{2^n-2} - 2^{2^{n-1}-1}, \tag{1}$$

where $\mathbf{S}(n)$ is the number of n-ary Sheffer functors. This formula has also been given without proof in [4].

In a recent paper [3], A. R. Turquette investigates the same problem and gives the answer

$$\mathbf{S}(n) = 2^{2^n-3} + 2^{2^n-4} + 2^{2^n-5} + \ldots + 2^{2^{n-1}-1}. \tag{2}$$

Having the well known equation for finite sums

$$\sum_{i=1}^{n} 2^{-i} = 1 - 2^{-n} \tag{3}$$

we can show the identity of the solutions (1) and (2). For from (2) we have

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$$\mathbf{S}(n) = \sum_{i=2}^{2^{n-1}} 2^{2^n - i - 1}$$

$$= 2^{2^n - 1} \sum_{i=2}^{2^{n-1}} 2^{-i}$$

And by (3) we get

$$\mathbf{S}(n) = 2^{2^n-1}[(1 - 2^{-2^{n-1}}) - (1 - \tfrac{1}{2})]$$

$$= 2^{2^n-1}[\tfrac{1}{2} - 2^{2^{n-1}-2^n}]$$

$$= 2^{2^n-2} - 2^{2^{n-1}-1}$$

which is our equation (1).

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