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A REDUCTION IN THE NUMBER OF INDEPENDENT AXIOM SCHEMATA FOR S4

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In "A Contribution to the Axiomatization of Lewis' System S5" [3], Sobociński has shown that S5 can be axiomatized with six mutually independent axioms and two rules of inference. In "New Axiomatizations of S3 and S4" [2], I had shown that S3 is axiomatizable with six mutually independent schemata and the Rule of detachment for material implication, and that the addition of the schematic analogue of Lewis' C10.1 results in an axiomatization of S4 for which the schemata are mutually independent under the Rule.

S4 can also be axiomatized with six schemata which are mutually independent under the Rule. To prove this assertion I use the apparatus of [2] and suppose its results at hand for reference. The schemata of the present system are *H*1-4, *H*6, and

H9: $\vdash \sim \Diamond \sim (\alpha \prec \Diamond \alpha)$.

The Rule is, of course, detachment for material implication. To show that S3 is included in the present system, it will suffice to derive *H*5 in it. *H*5 is not cited in the annotations to the proofs of Theorems 1-10 in [2]. Hence those results are obtainable in the present system, as is now *H*5.

*H*5: $\vdash (\alpha \prec \Diamond \alpha)$.

(1)	$\vdash [\sim \Diamond \sim (\alpha \prec \Diamond \alpha) \to \sim \sim (\alpha \prec \Diamond \alpha)]$	*H*4.
(2)	$\vdash \sim \Diamond \sim (\alpha \prec \Diamond \alpha)$	*H*9.
(3)	$\vdash \sim \sim (\alpha \prec \Diamond \alpha)$	(1) (2) Rule.
(4)	$\vdash [\sim \sim (\alpha \prec \Diamond \alpha) \prec (\alpha \prec \Diamond \alpha)]$	Th.7.
(5)	$\vdash (\alpha \prec \Diamond \alpha)$	(3) (4) Th.1.

Now, Lewis argues in effect ([1], p. 499) that the presence of a "necessarily-necessary" proposition in a system which includes S3 ensures that the system will contain C10. Besides S3, the present system contains the schematic analogue of a "necessarily-necessary" axiom, for it contains *H9*. By the remarks of Lewis ([1], p. 501), the present system contains S4.

As this result may be secured directly in the present system at not very great length, I give the relevant deductions. In view of Theorems 7

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and 8 and the Corollary to Theorem 21, either of the pair α, $\sim\sim\alpha$ can replace the other. This principle, Theorem 12, definitions, and iterated applications of Theorem 14 will be used tacitly.

Theorem 31. *If* $\vdash [\alpha \rightarrowtail (\beta \rightarrowtail \gamma)]$ *and* $\vdash (\alpha \rightarrowtail \beta)$, *then* $\vdash (\alpha \rightarrowtail \gamma)$.

(1) $\vdash [\alpha \rightarrowtail (\beta \rightarrowtail \gamma)]$ hyp.
(2) $\vdash (\alpha \rightarrowtail \beta)$ hyp.
(3) $\vdash \{(\alpha \wedge \alpha) \rightarrowtail [\beta \wedge (\beta \rightarrowtail \gamma)]\}$ (1) (2) Th.20.
(4) $\vdash [\alpha \rightarrowtail (\alpha \wedge \alpha)]$ *H*1.
(5) $\vdash \{[\beta \wedge (\beta \rightarrowtail \gamma)] \rightarrowtail \gamma\}$ Th.30.
(6) $\vdash (\alpha \rightarrowtail \gamma)$ (4) (3) (5) Th.14.

Theorem 32. $\vdash [\sim\Diamond\sim\alpha \rightarrowtail (\beta \rightarrowtail \alpha)]$.

(1) $\vdash [(\beta \wedge \sim\alpha) \rightarrowtail \sim\alpha]$ *H*2.
(2) $\vdash [\sim\Diamond\sim\alpha \rightarrowtail \sim\Diamond(\beta \wedge \sim\alpha)]$ (1) Th.2.

Theorem 33. *If* $\vdash [\alpha \rightarrowtail (\beta \rightarrowtail \gamma)]$ *and* $\vdash \sim\Diamond\sim\beta$, *then* $\vdash (\alpha \rightarrowtail \gamma)$.

(1) $\vdash [\alpha \rightarrowtail (\beta \rightarrowtail \gamma)]$ hyp.
(2) $\vdash \sim\Diamond\sim\beta$ hyp.
(3) $\vdash [\sim\Diamond\sim\beta \rightarrowtail (\alpha \rightarrowtail \beta)]$ Th.32.
(4) $\vdash (\alpha \rightarrowtail \beta)$ (2) (3) Th.1.
(5) $\vdash (\alpha \rightarrowtail \gamma)$ (1) (4) Th.31.

Theorem 34. $\vdash [\sim\Diamond\sim\alpha \rightarrowtail (\sim\Diamond\sim\beta \rightarrowtail \sim\Diamond\sim\alpha)]$.

(1) $\vdash [\sim\Diamond\sim\alpha \rightarrowtail (\beta \rightarrowtail \alpha)]$ Th.32.
(2) $\vdash [(\beta \rightarrowtail \alpha) \rightarrowtail (\sim\alpha \rightarrowtail \sim\beta)]$ Ths.12,10,21 (Cor.).
(3) $\vdash [(\sim\alpha \rightarrowtail \sim\beta) \rightarrowtail (\sim\Diamond\sim\beta \rightarrowtail \sim\Diamond\sim\alpha)]$ *H*6.
(4) $\vdash [\sim\Diamond\sim\alpha \rightarrowtail (\sim\Diamond\sim\beta \rightarrowtail \sim\Diamond\sim\alpha)]$ (1) (2) (3) Th.14.

Theorem 35. $\vdash [\sim\Diamond\sim\alpha \rightarrowtail (\sim\Diamond\sim\sim\Diamond\sim\beta \rightarrowtail \sim\Diamond\sim\sim\Diamond\sim\alpha)]$.

(1) $\vdash [\sim\Diamond\sim\alpha \rightarrowtail (\sim\Diamond\sim\beta \rightarrowtail \sim\Diamond\sim\alpha)]$ Th.34.
(2) $\vdash [(\sim\Diamond\sim\beta \rightarrowtail \sim\Diamond\sim\alpha) \rightarrowtail (\sim\sim\Diamond\sim\alpha \rightarrowtail \sim\sim\Diamond\sim\beta)]$ Ths.12,10,21 (Cor.).
(3) $\vdash [(\sim\sim\Diamond\sim\alpha \rightarrowtail \sim\sim\Diamond\sim\beta) \rightarrowtail (\sim\Diamond\sim\sim\Diamond\sim\beta \rightarrowtail \sim\Diamond\sim\sim\Diamond\sim\alpha)]$ *H*6.
(4) $\vdash [\sim\Diamond\sim\alpha \rightarrowtail (\sim\Diamond\sim\sim\Diamond\sim\beta \rightarrowtail \sim\Diamond\sim\sim\Diamond\sim\alpha)]$ (1) (2) (3) Th.14.

Theorem 36. $\vdash \sim\Diamond\sim\sim\Diamond\sim\sim\Diamond\sim(\alpha \rightarrow \Diamond\alpha)$.

(1) $\vdash \{(\alpha \rightarrowtail \Diamond\alpha) \rightarrowtail [\sim\Diamond\sim(\alpha \rightarrow \Diamond\alpha) \rightarrowtail \sim\Diamond\sim(\alpha \rightarrowtail \Diamond\alpha)]\}$ Th.34.
(2) $\vdash \sim\Diamond\sim(\alpha \rightarrowtail \Diamond\alpha)$ *H*9.
(3) $\vdash [\sim\Diamond\sim(\alpha \rightarrow \Diamond\alpha) \rightarrowtail \sim\Diamond\sim(\alpha \rightarrowtail \Diamond\alpha)]$ (1) (2) Th.1.
(4) $\vdash [\sim\sim\Diamond\sim(\alpha \rightarrowtail \Diamond\alpha) \rightarrowtail \sim\sim\Diamond\sim(\alpha \rightarrow \Diamond\alpha)]$ (3) Th.10.
(5) $\vdash [\sim\Diamond\sim\sim\Diamond\sim(\alpha \rightarrow \Diamond\alpha) \rightarrowtail \sim\Diamond\sim\sim\Diamond\sim(\alpha \rightarrowtail \Diamond\alpha)]$ (4) Th.2.
(6) $\vdash \sim\Diamond\sim\sim\Diamond\sim\sim\Diamond\sim(\alpha \rightarrow \Diamond\alpha)$ (2) (5) Th.1.

Theorem 37. $\vdash (\sim\Diamond\sim\alpha \rightarrowtail \sim\Diamond\sim\sim\Diamond\sim\alpha)$.

(1) $\vdash \{\sim\Diamond\sim\alpha \rightarrowtail [\sim\Diamond\sim\sim\Diamond\sim(\alpha \rightarrow \Diamond\alpha) \rightarrowtail \sim\Diamond\sim\sim\Diamond\sim\alpha]\}$ Th.34.

(2) $\vdash \sim \Diamond \sim \sim \Diamond \sim \sim \Diamond \sim (\alpha \rightarrow \Diamond \alpha)$ Th.36.

(3) $\vdash (\sim \Diamond \sim \alpha \rightarrowtail \sim \Diamond \sim \sim \Diamond \sim \alpha)$ (1) (2) Th.33.

Theorem 38. $\vdash (\Diamond \Diamond \alpha \rightarrowtail \Diamond \alpha)$.

(1) $\vdash (\sim \Diamond \sim \sim \alpha \rightarrowtail \sim \Diamond \sim \sim \Diamond \sim \sim \alpha)$ Th.37.

(2) $\vdash (\Diamond \Diamond \alpha \rightarrowtail \Diamond \alpha)$ (1) Th.9.

Theorem 38 is *H*7; hence the present system contains S4, by the result of [2]. By the same results, S4 contains the present system. In S4 we have

*H*9. $\vdash \sim \Diamond \sim (\alpha \rightarrowtail \Diamond \alpha)$.

(1) $\vdash [\Diamond \Diamond (\alpha \wedge \sim \Diamond \alpha) \rightarrowtail \Diamond (\alpha \wedge \sim \Diamond \alpha)]$ *H*7.

(2) $\vdash [\sim \Diamond (\alpha \wedge \sim \Diamond \alpha) \rightarrowtail \sim \Diamond \Diamond (\alpha \wedge \sim \Diamond \alpha)]$ (1) Th.10.

(3) $\vdash (\alpha \rightarrowtail \Diamond \alpha)$ *H*5.

(4) $\vdash \sim \Diamond \sim \sim \Diamond (\alpha \wedge \sim \Diamond \alpha)$ (2) (3) Th.1.

The following procedure will show that the schemata of the present axiomatization of S4 are mutually independent under the Rule of detachment for material implication. In the proof of mutual independence for the schemata of S3 in [2] replace every reference to *H*5 by a reference to *H*9.

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