THREE SET–THEORETICAL FORMULAS

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The set–theoretical formula which says that:

$\mathfrak{A}$. *If* $\mathfrak{m}$ *is a cardinal number which is not finite, then there exists no cardinal number* $\mathfrak{n}$ *such that* $\mathfrak{m} < \mathfrak{n} < 2^{\mathfrak{m}}$,

is called the generalized continuum hypothesis. It is known[1] that $\mathfrak{A}$ is inferentially equivalent to:

$\mathfrak{B}$. *The axiom of choice*

taken in conjunction with

$\mathfrak{C}$. *Cantor's hypothesis on alephs*

which says that

For any ordinal number α: $2^{\aleph_\alpha} = \aleph_{\alpha+1}$

Moreover, it is known[2] that $\mathfrak{C}$ is inferentially equivalent to:

$\mathfrak{D}$. If $\mathfrak{a}$ is an arbitrary aleph, then there exists no cardinal number such that $\mathfrak{a} < \mathfrak{n} < 2^{\mathfrak{a}}$.

The aim of this note is to show that the following three set–theoretical formulas:

A. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\mathfrak{n} < 2^{\mathfrak{m}}$, *then* $\mathfrak{n} \leqslant \mathfrak{m}$.[3]

B. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\mathfrak{n} < 2^{\mathfrak{m}}$, *then either* $\mathfrak{n} \leqslant \mathfrak{m}$ *or* $\mathfrak{m} < \mathfrak{n}$.

C. *For any cardinal number* $\mathfrak{n}$ *which is not finite and any cardinal number* $\mathfrak{a}$, *if* $\mathfrak{a}$ *is an aleph and* $\mathfrak{n} < 2^{\mathfrak{a}}$, *then* $\mathfrak{n} \leqslant \mathfrak{a}$.

are such that formula **A** is equivalent to $\mathfrak{A}$, formula **B**–to $\mathfrak{B}$ and formula **C**–to $\mathfrak{C}$. It seems to me that this fact, which as far as I know has not been noticed, is of some interest, because the formulas **A**, **B**, and **C** having very similar structure elucidate the mutual connections among the fundamental laws $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$.

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Besides, it will be proved that $\mathfrak{E}$ is inferentially equivalent to the conjunction of the following two formulas:

E$_1$. *For any cardinal numbers* $\mathfrak{a}$ *and* $\mathfrak{b}$, *if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are alephs and* $\mathfrak{b} < 2^{\mathfrak{a}}$, *then* $\mathfrak{b} \leqslant \mathfrak{a}$.

E$_2$. *For any cardinal numbers* $\mathfrak{a}$ *and* $\mathfrak{b}$, *if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are alephs and* $\mathfrak{b} < 2^{2^{\mathfrak{a}}}$, *then* $\mathfrak{b} \leqslant 2^{\mathfrak{a}}$.

Proof:[4]

(i) *Formula* $\mathfrak{A}$ *implies* **A**. We know that formulas $\mathfrak{B}$ and $\mathfrak{C}$ follow from $\mathfrak{A}$. Assume the conditions of **A**, viz. that $\mathfrak{m}$ and $\mathfrak{n}$ are arbitrary cardinal numbers which are not finite and that $\mathfrak{n} < 2^{\mathfrak{m}}$. Hence, in virtue of the axiom of choice ($\mathfrak{B}$), $\mathfrak{m}$, $\mathfrak{n}$ and $2^{\mathfrak{m}}$ are alephs. Therefore, there are ordinal numbers α, β, and γ such that

1. $\mathfrak{m} = \aleph_\alpha$; $\mathfrak{n} = \aleph_\beta$; $2^{\mathfrak{m}} = \aleph_\gamma$

Since $\mathfrak{n} < 2^{\mathfrak{m}}$, then, in virtue of *1*, we have $\aleph_\beta < \aleph_\gamma$. Hence:

2. $\beta < \gamma$

On the other hand it follows from *1* and Cantor's hypothesis on alephs ($\mathfrak{C}$) that:

3. $\aleph_\gamma = 2^{\mathfrak{m}} = 2^{\aleph_\alpha} = \aleph_{\alpha+1}$

and, therefore, that $\gamma = \alpha + 1$, i.e. that γ is the next ordinal number to α. This fact and point *2* allow us to establish that *it is not true that* $\alpha < \beta$, because $\beta < \gamma = \alpha + 1$. Hence:

4. *It is not true that* $\aleph_\alpha < \aleph_\beta$

Since, obviously, for alephs we have *either* $\aleph_\beta \leqslant \aleph_\alpha$ *or* $\aleph_\alpha < \aleph_\beta$, then points *4* and *1* at once imply that

$\mathfrak{n} \leqslant \mathfrak{m}$

and, therefore, that formula **A** is a consequence of $\mathfrak{A}$.

(ii) *Formula* $\mathfrak{B}$ *implies* **B**. This is obvious, since the axiom of choice is equivalent to the law of trichotomy for cardinals and **B** is nothing else than a weak formulation of this law, obtainable by simple application of the propositional calculus.

(iii) *Formula* $\mathfrak{E}$ *implies* **C**. Assume the conditions of **C**, viz. that $\mathfrak{n}$ is an arbitrary cardinal number which is not finite, $\mathfrak{a}$ is an aleph and that $\mathfrak{n} < 2^{\mathfrak{a}}$. Since $\mathfrak{a}$ is an aleph, then there is an ordinal number α such that

5. $\mathfrak{a} = \aleph_\alpha$

Since $\mathfrak{n} < 2^{\mathfrak{a}}$, then in virtue of *5* and $\mathfrak{E}$ we have:

6. $\mathfrak{n} < 2^{\mathfrak{a}} = 2^{\aleph_\alpha} = \aleph_{\alpha+1}$

which shows that $\mathfrak{n}$ and $2^{\mathfrak{a}}$ are alephs and, moreover, that $2^{\mathfrak{a}}$ is the next aleph to $\mathfrak{a}$. Hence, from the law of trichotomy for alephs and *6* it follows that the case $\mathfrak{a} < \mathfrak{n} < 2^{\mathfrak{a}}$ is impossible, and that we must have

$\mathfrak{n} \leqslant \mathfrak{a}$.

Thus, formula **C** follows from $\mathfrak{C}$.

(iv) *Formula* $\mathfrak{C}$ *implies* $\mathfrak{D}$. Assume the condition of $\mathfrak{D}$, viz. that $\mathfrak{a}$ is an arbitrary aleph. Hence, there is an ordinal number α such that $\mathfrak{a} = \aleph_\alpha$ which in virtue of $\mathfrak{C}$ gives

7. $2^{\mathfrak{a}} = 2^{\aleph_\alpha} = \aleph_{\alpha+1}$

It shows that $2^{\mathfrak{a}}$ is the next aleph to $\mathfrak{a}$ and, therefore,

there exists no cardinal number $\mathfrak{n}$ *such that* $\mathfrak{a} < \mathfrak{n} < 2^{\mathfrak{a}}$,

Thus, formula $\mathfrak{D}$ is a consequence of $\mathfrak{C}$.

(v) *Formula* $\mathfrak{D}$ *implies* $\mathbf{E}_1$ *and* $\mathbf{E}_2$. Assume the conditions of $\mathbf{E}_1$, viz. that $\mathfrak{a}$ and $\mathfrak{b}$ are arbitrary alephs and that $\mathfrak{b} < 2^{\mathfrak{a}}$. Since the law of trichotomy holds for any aleph, we have *either* $\mathfrak{b} \leqslant \mathfrak{a}$ *or* $\mathfrak{a} < \mathfrak{b}$. But, the case that $\mathfrak{a} < \mathfrak{b}$ is impossible, since in connection with the fact that $\mathfrak{b} < 2^{\mathfrak{a}}$ it yields $\mathfrak{a} < \mathfrak{b} < 2^{\mathfrak{a}}$, which is excluded by $\mathfrak{D}$. Therefore, it must be that

$\mathfrak{b} \leqslant \mathfrak{a}$

which proves that $\mathbf{E}_1$ follows from $\mathfrak{D}$.

The proof of $\mathbf{E}_2$ requires the following lemma:

Lemma I. *For any cardinal numbers* $\mathfrak{a}$ *and* $\mathfrak{b}$, *if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are alephs,* $\mathfrak{b} < 2^{2^{\mathfrak{a}}}$ *and* $\mathfrak{a} < \mathfrak{b}$, *then* $\mathfrak{b} = 2^{\mathfrak{a}}$.

In order to prove that this lemma follows from $\mathfrak{D}$ assume its conditions, viz. that $\mathfrak{a}$ and $\mathfrak{b}$ are arbitrary alephs, $\mathfrak{b} < 2^{2^{\mathfrak{a}}}$ and that $\mathfrak{a} < \mathfrak{b}$. The condition $\mathfrak{a} < \mathfrak{b}$ at once implies (without the use of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ or $\mathfrak{D}$) that $2^{\mathfrak{a}} \leqslant 2^{\mathfrak{b}}$. But the case $2^{\mathfrak{a}} = 2^{\mathfrak{b}}$ is impossible, because together with our assump- that $\mathfrak{a}$ is an aleph and $\mathfrak{a} < \mathfrak{b}$, and general formula $\mathfrak{b} < 2^{\mathfrak{b}}$ it give $\mathfrak{a} < \mathfrak{b} < 2^{\mathfrak{b}} = 2^{\mathfrak{a}}$ which is excluded by $\mathfrak{D}$. Hence, it must be that

8. $2^{\mathfrak{a}} < 2^{\mathfrak{b}}$.

Since $\mathfrak{b}$ is an aleph, obviously, we have $2^{\mathfrak{b}} = 2^{\mathfrak{b}+1} = 2^{\mathfrak{b}} + 2^{\mathfrak{b}}$. Therefore, it follows from *8* and the general formula $\mathfrak{b} < 2^{\mathfrak{b}}$ that:

9. $2^{\mathfrak{a}} + \mathfrak{b} \leqslant 2^{\mathfrak{b}} + 2^{\mathfrak{b}} = 2^{\mathfrak{b}}$.

In virtue of the well known theorem[5], which says that

$\mathfrak{F}$. *For any cardinal number* $\mathfrak{m}$ *such that* $\mathfrak{m} \geqslant \aleph_0$: $2^{\mathfrak{m}} - \mathfrak{m} = 2^{\mathfrak{m}}$

and which is provable without the aid of the axiom of choice, and the obvious fact that $\mathfrak{b} \geqslant \aleph_0$, we can establish that there exists one and only one cardinal number $\mathfrak{p}$ such that $\mathfrak{b} + \mathfrak{p} = 2^{\mathfrak{b}}$, namely: $\mathfrak{p} = 2^{\mathfrak{b}}$. This excludes the case $2^{\mathfrak{a}} + \mathfrak{b} = 2^{\mathfrak{b}}$ of 9, since the possibility that $2^{\mathfrak{a}} = 2^{\mathfrak{b}}$ is already rejected. Hence, it must be that $2^{\mathfrak{a}} + \mathfrak{b} < 2^{\mathfrak{b}}$, which at once implies that

10. $\quad \mathfrak{b} \leqslant 2^{\mathfrak{a}} + \mathfrak{b} < 2^{\mathfrak{b}}$

But, in virtue of $\mathfrak{D}$ and the fact that $\mathfrak{b}$ is an aleph, the case that $\mathfrak{b} < 2^{\mathfrak{a}} + \mathfrak{b} < 2^{\mathfrak{b}}$ is excluded. Hence, it must be that $\mathfrak{b} = 2^{\mathfrak{a}} + \mathfrak{b}$, which shows that

11. $\quad 2^{\mathfrak{a}} \leqslant \mathfrak{b}$.

Hence, the assumptions that $\mathfrak{b}$ is an aleph and $\mathfrak{b} < 2^{2^{\mathfrak{a}}}$, and the point *11* imply that $2^{\mathfrak{a}} \leqslant \mathfrak{b} < 2^{2^{\mathfrak{a}}}$ and, moreover, that $2^{\mathfrak{a}}$ is an aleph. From this last conclusion and $\mathfrak{D}$ it follows immediately that the case $2^{\mathfrak{a}} < \mathfrak{b} < 2^{2^{\mathfrak{a}}}$ is excluded. Therefore, we have:

$$\mathfrak{b} = 2^{\mathfrak{a}}$$

which shows that *Lemma* I follows from $\mathfrak{D}$.

Now, assume the conditions of $\mathbf{E}_2$, viz. that $\mathfrak{a}$ and $\mathfrak{b}$ are arbitrary alephs and that $\mathfrak{b} < 2^{2^{\mathfrak{a}}}$. Since due to the law of trichotomy for alephs we have *either* $\mathfrak{b} \leqslant \mathfrak{a}$ *or* $\mathfrak{a} < \mathfrak{b}$, then our assumptions and *Lemma* I allow us to establish the following alternative:

12. $\quad$ *either* $\mathfrak{b} \leqslant \mathfrak{a}$ *or* $\mathfrak{b} = 2^{\mathfrak{a}}$

which in virtue of the general formula $\mathfrak{a} < 2^{\mathfrak{a}}$ gives

$$\mathfrak{b} \leqslant 2^{\mathfrak{a}}$$

Thus, $\mathbf{E}_2$ is a consequence of $\mathfrak{D}$.

(vi) *Formula* **A** *implies* $\mathfrak{A}$. Assume the condition of $\mathfrak{A}$, viz. that $\mathfrak{m}$ is a cardinal number which is not finite, and let us suppose that there is a cardinal number $\mathfrak{n}$ such that $\mathfrak{m} < \mathfrak{n} < 2^{\mathfrak{m}}$. Since $\mathfrak{m} < \mathfrak{n}$ and $\mathfrak{m}$ is not finite by the assumption, $\mathfrak{n}$ must be not finite. Hence, in virtue of **A**, our supposition yields that $\mathfrak{n} \leqslant \mathfrak{m}$, which shows that it implies a contradiction. Therefore, its negation is true, viz.:

There exists no cardinal number $\mathfrak{n}$ *such that* $\mathfrak{m} < \mathfrak{n} < 2^{\mathfrak{m}}$.

which shows that $\mathfrak{A}$ follows from **A**.

(vii) *Formula* **B** *implies* $\mathfrak{B}$. In order to prove that the axiom of choice ($\mathfrak{B}$) follows from **B**, it is, evidently, sufficient to show that **B** implies that any arbitrary cardinal number which is not finite is an aleph. Assume, therefore, that $\mathfrak{m}$ is an arbitrary cardinal number which is not finite. It is well known[6], that without the aid of the axiom of choice we can associate with $\mathfrak{m}$ a certain so-called Hartogs' aleph, viz. $\aleph(\mathfrak{m})$, which possesses the following properties:

13. $\quad \aleph(\mathfrak{m})$ *is the least aleph such that* $\aleph(\mathfrak{m})$ *is not* $\leqslant \mathfrak{m}$ *and* $\aleph(\mathfrak{m}) \leqslant 2^{2^{2^{\mathfrak{m}}}}$.

The case $\aleph(\mathfrak{m}) = 2^{2^{2^{\mathfrak{m}}}}$ gives at once $\aleph(\mathfrak{m}) > \mathfrak{m}$, i.e. that $\mathfrak{m}$ is an aleph. Assume, therefore, that $\aleph(\mathfrak{m}) < 2^{2^{2^{\mathfrak{m}}}}$. Since $\mathfrak{m}$ is not finite, $2^{2^{\mathfrak{m}}}$ is not finite either. Hence, in virtue of **B** ($\mathfrak{m} = 2^{2^{\mathfrak{m}}}$ and $\mathfrak{n} = \aleph(\mathfrak{m})$) we obtain

14. *either* $\aleph(\mathfrak{m}) \leqslant 2^{2^{\mathfrak{m}}}$ *or* $2^{2^{\mathfrak{m}}} < \aleph(\mathfrak{m})$

But, the cases $\aleph(\mathfrak{m}) = 2^{2^{\mathfrak{m}}}$ and $2^{2^{\mathfrak{m}}} < \aleph(\mathfrak{m})$ of *14* imply that $\aleph(\mathfrak{m}) > \mathfrak{m}$, i.e. that $\mathfrak{m}$ is an aleph. Hence, it remains to investigate the third case, viz. that $\aleph(\mathfrak{m}) < 2^{2^{\mathfrak{m}}}$. From the fact that $\mathfrak{m}$ is not finite it follows that $2^{\mathfrak{m}}$ possesses the same property. Hence, in virtue of **B** ($\mathfrak{m} = 2^{\mathfrak{m}}$ and $\mathfrak{n} = \aleph(\mathfrak{m})$) we conclude that

15. *either* $\aleph(\mathfrak{m}) \leqslant 2^{\mathfrak{m}}$ *or* $2^{\mathfrak{m}} < \aleph(\mathfrak{m})$

Since the cases $\aleph(\mathfrak{m}) = 2^{\mathfrak{m}}$ and $2^{\mathfrak{m}} < \aleph(\mathfrak{m})$ give $\aleph(\mathfrak{m}) > \mathfrak{m}$, i.e. that $\mathfrak{m}$ is an aleph, only the third case, viz. $\aleph(\mathfrak{m}) < 2^{\mathfrak{m}}$, has to be analyzed. But this case and **B** ($\mathfrak{n} = \aleph(\mathfrak{m})$) imply that

16. *either* $\aleph(\mathfrak{m}) \leqslant \mathfrak{m}$ *or* $\mathfrak{m} < \aleph(\mathfrak{m})$

But the first possibility of *16* is excluded by the properties of Hartogs' alephs given in *13*. Therefore, we have $\aleph(\mathfrak{m}) > \mathfrak{m}$, i.e. that $\mathfrak{m}$ is an aleph. This concludes the proof, since the points *14*, *15* and *16* show that in virtue of **B** our arbitrary cardinal number $\mathfrak{m}$ which is not finite is an aleph.

Thus, $\mathfrak{B}$ follows from **B**.

(viii) *Formula* **C** *implies* $\mathfrak{D}$. Assume the condition of $\mathfrak{D}$, viz. that $\mathfrak{a}$ is an arbitrary aleph, and let us suppose that there is a cardinal number $\mathfrak{n}$ such that $\mathfrak{a} < \mathfrak{n} < 2^{\mathfrak{a}}$. Since $\mathfrak{a}$ is an aleph and $\mathfrak{a} < \mathfrak{n}$, then $\mathfrak{n}$ is not finite. Therefore, our supposition and **C** give a contradiction, since they imply $\mathfrak{n} \leqslant \mathfrak{a}$. Hence:

there exists no cardinal number $\mathfrak{n}$ *such that* $\mathfrak{a} < \mathfrak{n} < 2^{\mathfrak{a}}$.

which shows that $\mathfrak{D}$ follows from **C**.

(ix) *Formulas* $\mathbf{E}_1$ *and* $\mathbf{E}_2$ *imply* $\mathfrak{E}$. Assume the condition of $\mathfrak{E}$, viz. that α is an arbitrary ordinal number. Hence, the aleph $\aleph_\alpha$ exists and since $\aleph_\alpha$ is, obviously, a cardinal number which is not finite, we can associate with it a Hartogs' aleph $\aleph(\aleph_\alpha)$, i.e. an aleph which possesses the following properties:[6]

17. $\aleph(\aleph_\alpha)$ *is the least aleph such that* $\aleph(\aleph_\alpha)$ *is not* $\leqslant \aleph_\alpha$ *and* $\aleph_\alpha(\aleph_\alpha) \leqslant 2^{2^{\aleph_\alpha^2}}$

and this can be established without the aid of the axiom of choice. It is known[6] that the least Hartogs' aleph for any $\aleph_\beta$ is $\aleph_{\beta+1}$. Hence, we have $\aleph(\aleph_\alpha) = \aleph_{\alpha+1}$ and since $\aleph_\alpha = \aleph_\alpha^2$, we can conclude from *17* that

18. $\aleph_{\alpha+1} \leqslant 2^{2^{\aleph_\alpha^2}} = 2^{2^{\aleph_\alpha}}$

The case $\aleph_{\alpha+1} = 2^{2^{\aleph_\alpha}}$ of *18* is excluded, since it implies $\aleph_\alpha < 2^{\aleph_\alpha} < \aleph_{\alpha+1}$, which is impossible. Hence, the formula $\aleph_{\alpha+1} < 2^{2^{\aleph_\alpha}}$ holds, which in virtue of $\mathbf{E}_2$ ($\alpha = \aleph_\alpha$ and $\mathfrak{b} = \aleph_{\alpha+1}$) gives

19. $\aleph_{\alpha+1} \leqslant 2^{\aleph}$

The case $\aleph_{\alpha+1} < 2^{\aleph_\alpha}$ is impossible, since in virtue of $\mathbf{E}_1$ ($\mathfrak{a} = \aleph_\alpha$ and $\mathfrak{b} = \aleph_{\alpha+1}$) it implies $\aleph_{\alpha+1} \leqslant \aleph_\alpha$. Hence, the second case of *19* is true, viz. that

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha}$$

Thus, $\mathfrak{C}$ follows from $\mathbf{E}_1$ and $\mathbf{E}_2$.

This concludes the required proofs, since:[7] 1) (i) and (v̈i) show that $\{\mathfrak{A}\} \rightleftarrows \{\mathbf{A}\}$; 2) (ii) and (v̈ii) – that $\{\mathfrak{B}\} \rightleftarrows \{\mathbf{B}\}$; 3) (iii), (iv̈) and (v̈) – that $\{\mathfrak{C}\} \rightarrow \{\mathbf{C}\}$; $\{\mathfrak{C}\} \rightarrow \{\mathfrak{D}\}$ and $\{\mathfrak{D}\} \rightarrow \{\mathbf{E}_1; \mathbf{E}_2\}$; 4) (v̈iii) and (iẍ) – that $\{\mathbf{C}\} \rightarrow \{\mathfrak{D}\}$ and $\{\mathbf{E}_1; \mathbf{E}_2\} \rightarrow \{\mathfrak{C}\}$. Therefore, evidently: 5) $\{\mathfrak{C}\} \rightleftarrows \{\mathbf{C}\}$; 6) $\{\mathfrak{C}\} \rightleftarrows \{\mathbf{E}_1; \mathbf{E}_2\}$ and, incidentally, an already known fact, viz. that 7) $\{\mathfrak{C}\} \rightleftarrows \{\mathfrak{D}\}$, is confirmed.

It has to be noted that a simple inspection of the proofs given in (i), (v̈i), (ii) and (v̈ii) shows that **A** is inferentially equivalent to:

A*. *For any cardinal number* $\mathfrak{m}$ *which is not finite and any cardinal number* $\mathfrak{a}$, *if* $\mathfrak{a}$ *is an aleph and* $\mathfrak{a} < 2^{\mathfrak{m}}$; *then* $\mathfrak{a} \leqslant \mathfrak{m}$.

and that **B** is inferentially equivalent to:

B*. *For any cardinal number* $\mathfrak{m}$ *which is not finite and any cardinal number* $\mathfrak{a}$, *if* $\mathfrak{a}$ *is an aleph and* $\mathfrak{a} < 2^{\mathfrak{m}}$; *then either* $\mathfrak{a} \leqslant \mathfrak{m}$ *or* $\mathfrak{m} < \mathfrak{a}$.

Applying the reasonings of (vii) to **B*** we obtain $\mathfrak{B}$ and, therefore, **B**. Each formula **A** and **A*** implies **B** and under this condition there is no difficulty to show that **A** follows from **A***.

It is worth – while to note the similarity of the formulas **A*** and **C**. I do not know whether $\mathbf{E}_2$ (or $\mathbf{E}_1$) taken alone implies $\mathfrak{D}$. It is easy to show that $\{\mathfrak{B}; \mathbf{E}_1\} \rightleftarrows \{\mathbf{A}\}$ and, therefore, $\{\mathfrak{B}; \mathbf{E}_1\} \rightleftarrows \{\mathfrak{A}\}$.

NOTES

1. This was announced without proof by Lindenbaum and Tarski, *cf.* [3], pp. 313 – 314, theorem **95**. Sierpiński published a proof in [4], pp. 434 – 437. *Cf.* also his [7] and [6], pp. 166 – 167 and pp. 193 – 197. *Cf.* also [2], pp. 245 – 247. In [3], [6] and [7] the condition of $\mathfrak{A}$ is stronger, viz. $\mathfrak{m}$ is assumed to be a transfinite cardinal, i.e. $\mathfrak{m} \geqslant \aleph_0$. This assumption is superfluous, *cf.* [4], pp. 434 – 437.

2. This was announced without proof by the authors in [3], p. 314, theorem **96**. As far as I know the proof was never published. The deductions presented in this paper prove by the way that $\mathfrak{C}$ is equivalent to $\mathfrak{D}$.
3. In [3], p. 314, the authors announced the following definitions:
   85. *Le nombre cardinal* $\mathfrak{m}$ *jouit de la propriété P, lorsque aucun nombre* $\mathfrak{r}$ *ne satisfait à la formule* $\mathfrak{m} < \mathfrak{r} < 2^{\mathfrak{m}}$.
   and theorem (without proof):
   87. *Si* $\mathfrak{m}$ *jouit de la propriété P et* $\mathfrak{n} < 2^{\mathfrak{m}}$, *on a* $\mathfrak{n} \leqslant \mathfrak{m}$.
   Perhaps, the authors had noticed that $\mathfrak{A}$ implies **A**.
4. The proofs which are given below are established within the general set theory, i.e. the set theory from which the assumptions $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and all their consequences otherwise unprovable have been removed. It is well known that if we base so defined a general set theory on an axiomatic system in which the notions of the cardinal and ordinal numbers cannot be defined, we have to introduce these concepts into the system by means of special axioms.
5. This theorem is announced without proof by Tarski in [3], p. 307, theorem **56**. *Cf.* also *Annales de la Société Polonaise de Mathématique*, v. 5 (1926), p. 101. A proof is given by Sierpiński in [7] and [4], pp. 168 – 170. *Cf.* also [2], p. 118. Concerning the definition of the difference of two cardinal numbers, *cf.* e.g. [3], p. 305, definition **47** and [4], p. 156.
6. In [1], Hartogs proved that with any cardinal number $\mathfrak{m}$ which is not finite we can associate an aleph $\aleph(\mathfrak{m})$ such that $\aleph(\mathfrak{m})$ is not $\leqslant \mathfrak{m}$. Later, it was established that if $\aleph(\mathfrak{m})$ is the least such aleph in respect to $\mathfrak{m}$, then: 1) $\aleph(\mathfrak{m}) \leqslant 2^{2^{2^{\mathfrak{m}}}}$ and in the same time $\aleph(\mathfrak{m}) \leqslant 2^{2^{\mathfrak{m}^2}}$; 2) if $\mathfrak{m} = \aleph_\beta$, then $\aleph(\mathfrak{m}) = \aleph_{\beta+1}$. *Cf.* [3], pp. 311 – 312, [4], p. 407 – 409 and [2], pp. 220 – 221. The existence of Hartogs' aleph and its properties are provable without the aid of the axiom of choice.
7. The symbols $\rightleftarrows$ and $\rightarrow$ mean: "is inferentially equivalent" and "implies" respectively.

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