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A SEMI-LATTICE THEORETICAL CHARACTERIZATION OF ASSOCIATIVE NEWMAN ALGEBRAS

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The aim of this note[1] is to stress a fact which, due to the original formulation of Newman's systems given in [1], can be easily overlooked: an associative Newman algebra can be considered as a semi-lattice with respect to the binary operation $\times$ to which the additional postulates are added concerning the properties of the binary operation + (which is neither a lattice-theoretical join nor a lattice theoretical symmetrical difference) and the unary operation -, i.e., the complementation peculiar to this system. Namely, it will be shown that in the field of the axioms *A1-A11* the proper axioms of system $\mathfrak{D}$ of associative Newman algebra, *cf.* [2], section **3**, i.e., the postulates

$F1 \quad [ab]:a,b\in B\,.\supset.\,a = a + (b\times\bar{b})$
$F2 \quad [ab]:a,b\in B\,.\supset.\,a = a\times(b+\bar{b})$
$H1 \quad [abc]:a,b,c\in B\,.\supset.\,a\times(b+c) = (c\times a)+(b\times a)$
$L1 \quad [abc]:a,b,c\in b\,.\supset.\,a\times(b\times c) = (a\times b)\times c$

are inferentially equivalent to the following formulas: *F1*, *F2*, *L1* and

$F33 \quad [ab]:a,b\in B\,.\supset.\,a\times b = b\times a$
$C1 \quad [abc]:a,b,c\in B\,.\supset.\,a\times(b+c) = (a\times b)+(a\times c)$

and, moreover, that the idempotent law with respect to operation $\times$, i.e.,

$F7 \quad [a]:a\in B\,.\supset.\,a = (a\times a)$

is a consequence of the axioms *F1*, *F2* and *C1*.

Proof: In [2], section **3**, it has been proved that the formulas *F33* and *C1* follow from *F1*, *F2*, *H1* and *L1*. On the other hand, let us assume *F1*, *F2*, *L1*, *F33* and *C1*. Then:

1. An acquaintance with the papers [1], [2] and [3] is presupposed. An enumeration of the formulas discussed in this note is the same which they have in [3] and [2]. The axioms *A1-A11*, *cf.* [3], section 1, will be used tacitly in the deductions presented in this note.

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C2 $[abc]:a,b,c\in B\,.\supset.\,(a+b)\times c=(a\times c)+(b\times c)$ [*F33*; *C1*]

F3 $[ab]:a,b\in B\,.\supset.\,a=(b+\overline{b})\times a$ [*F1*; *F33*]

K1 $[ab]:a,b\in B\,.\,a+a=a\times\overline{a}\,.\,b+b=b\times\overline{b}\,.\supset.\,a\times(b\times b)=(a\times b)\times b$ [*L1*]

F7 $[a]:a\in B\,.\supset.\,a=a\times a$

PR $[a]:\mathrm{Hp}(1)\,.\supset.$

$a=a\times(a+\overline{a})=(a\times\overline{a})+(a\times a)=a\times a$ [1; *F2*; *C1*; *F1*]

Since, *cf.* [2], section **2**, it has been proved that $\{C1; C2; F1; F2; F3; K1\}\rightarrow\{F1; F2; H1; L1\}$, we have $\{F1; F2; H1; L1\}\rightleftarrows\{F1; F2; L1; F33; C1\}$. And, moreover, it is shown that *F1*, *F2* and *C1* imply *F7*. Thus, the proof is complete.

The mutual independence of the axioms *F1*, *F2*, *L1*, *F33* and *C1* is established by using Newman's example E10, *cf.* [1], p. 271, matrices $\mathfrak{M}3$ and $\mathfrak{M}4$, *cf.* [3], section 4, and the following algebraic tables (matrices):

$\mathfrak{M}7$

+	0	α	β	γ	δ
0	0	α	β	γ	δ
α	α	α	δ	δ	δ
β	β	δ	β	δ	δ
γ	γ	δ	δ	γ	δ
δ	δ	δ	δ	δ	δ

×	0	α	β	γ	δ
0	0	0	0	0	0
α	0	α	0	0	α
β	0	0	β	0	β
γ	0	0	0	γ	γ
δ	0	α	β	γ	δ

x	$\overline{x}$
0	δ
α	β
β	γ
γ	α
δ	0

and

$\mathfrak{M}8$

+	0	α	β	γ	δ
0	0	α	β	γ	δ
α	α	α	δ	δ	δ
β	β	δ	β	β	δ
γ	γ	δ	β	γ	δ
δ	δ	δ	δ	δ	δ

×	0	α	β	γ	δ
0	0	0	0	0	0
α	0	α	0	0	α
β	0	0	β	β	β
γ	0	0	γ	γ	γ
δ	0	α	β	γ	δ

x	$\overline{x}$
0	δ
α	β
β	α
γ	α
δ	0

Matrices $\mathfrak{M}7$ and $\mathfrak{M}8$ are the examples KP_1 and E1 of Stone, *cf.* [4], p. 731, and Newman, *cf.* [1], p. 268, respectively, but adjusted to the primitive unary operation - of system $\mathfrak{D}$. Since:

(a) example E10 verifies all postulates of (non-associative) Newman algebra, but falsifies *L1*, and the formulas *F1*, *F2*, *F33* and *C1* are provable in the field of that system, *cf.* [2] and [3],
(b) matrix $\mathfrak{M}3$ verifies *F2*, *L1*, *F33* and *C1*, but falsifies *F1*, *cf.* [3], section **4**,
(c) matrix $\mathfrak{M}4$ verifies *F1*, *L1*, *F33* and *C1*, but falsifies *F2*, *cf.* [3], section **4**,
(d) matrix $\mathfrak{M}7$ verifies *F1*, *F2*, *L1* and *F33*, but falsifies *C1* for a/α, b/β and c/γ: (i) $\alpha\times(\beta+\gamma)=\alpha\times\delta=\alpha$, and (ii) $(\alpha\times\beta)+(\alpha\times\gamma)=0+0=0$,
(e) matrix $\mathfrak{M}8$ verifies *F1*, *F2*, *L1* and *C1*, but falsifies *F33* for a/β and b/γ: (i) $\beta\times\gamma=\beta$, and (ii) $\gamma\times\beta=\gamma$,

we know that the axioms *F1*, *F2*, *L1*, *F33* and *C1* are mutually independent.

REMARK: Although it is established above that associative Newman algebra can be considered as a semi-lattice with respect to the operation $\times$, the axiom-system $\{F1; F2; L1; F33; C1\}$ does not contain $F7$, i.e., the idempotent law for $\times$, as an independent axiom. If for some reasons it would be desired to have an axiom-system of this algebra such that its axioms would be mutually independent, and that it would contain $F7$, $F33$ and $L1$, such axiomatization can be constructed as follows: Assume, as the axioms, $F7$, $F33$, $L1$ and instead of $F1$, $F2$ and $C1$ the formulas

$F1^*$ $[ab]:a,b\epsilon B\,.\supset.\,a\times a = a + (b\times \bar{b})$
$F2^*$ $[ab]:a,b\epsilon B\,.\supset.\,a\times (a\times a) = a\times (b + \bar{b})$
$C1^*$ $[abc]:a,b,c\epsilon B\,.\supset.\,(a\times ((a\times a)\times (a\times a)))\times (b + c) = (a\times b) + (a\times c)$

It is self-evident that the axioms $F7$, $F33$, $L1$, $F1^*$, $F2^*$ and $C1^*$ are mutually independent, and that $\{F1; F2; L1; F33; C1\} \rightleftarrows \{F7; F33; L11; F1^*; F2^*; C1^*\}$. I was unable to construct a more natural axiom-system possessing the required property.

REFERENCES

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