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AN APPLICATION OF MATHEMATICAL LOGIC TO THE INTEGER LINEAR PROGRAMMING PROBLEM

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The general integer linear programming problem, namely optimising a linear function subject to non-negative integer solutions of a set of simultaneous linear inequalities, was until 1958 one of the major unsolved problems in the theory of linear programming. Yet, as I shall show in this paper, a technique for solving this problem was available in 1929 in the paper by M. Presburger [1] which provided a decision procedure for a certain fragment of recursive arithmetic. Of course it must be said that the subject of linear programming did not exist in 1929 and so Presburger's algorithm was in this context a solution to a non-existent problem. I shall describe briefly Presburger's result and then show that the existence of an algorithm for solving the integer linear programming problem is an obvious consequence of it. I make no claims for the algorithm as a practical means of computation when compared with the solution due to R. E. Gomory in 1958 [2], or any subsequent one. It is however a simple method of showing that the problem is soluble and predates Gomory's solution by almost 30 years.

Presburger provided a decision procedure (i.e. a procedure for deciding whether statements were true or false) for the formal system of arithmetic referred to as system D in Hilbert and Bernays Vol. I [3]. System D refers to the 1st order theory with the one predicate, equality, one constant, 0, and two functions, S (the successor function) and the addition function. By applying the function S to the constant 0 we have all the natural numbers in the system, and applying the addition function to the variables and constants we obtain as the terms of the system, linear forms in any number of variables with positive coefficients and constants. The atomic formulae of the system are the expressions $s = t$ where s, t are terms, and hence the atomic formulae are just linear equations with the variables on their positive side.

Thus the system contains the propositional connectives for "not"

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"and" and "or", and the usual two quantifiers (Ex) for "there exists x" and (x) for "for all x", so the formulae which Presburger's procedure enables one to decide the truth or falsity of are, conjunctions and disjunctions of linear equations and inequalities involving the two quantifiers for "there exists" and "for all . . .", e.g.:

$$(\mathrm{E}x)\ (y)\ (((2x = y + 3) \wedge (2x + 2y = 11)) \vee (\mathrm{E}z)\ (2x + 3y = z))$$

The procedure consists in successively replacing given formulae by equivalent formulae containing one less variable, until such time as no variables remain and we are left with a set of relations between constants, the truth or falsity of which can easily be decided.

The integer linear programming problem is to optimise a linear form

$$c_1x_1 + c_2x_2 \ldots\ldots + c_nx_n \tag{1}$$

subject to non-negative integer solutions of a set of simultaneous inequalities

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 \ldots\ldots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ \ldots\ + a_{2n}x_n \leq b_2 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_n + \ \ldots\ a_{mn}x_n \leq b_m \end{array} \tag{2}$$

where a_{ij}, b_i are integers, positive negative or zero.

For purposes of this part we shall assume that the inequalities have been made into equations by the use of slack variables and further that each variable and the constants have been transferred to that side on which their coefficients (or the constants) are positive. Thus we shall replace (2) by equations

$$\begin{array}{c} A_1(x_1 \ldots\ldots x_t) = B_1(x_1 \ldots\ldots x_t) \\ \vdots \qquad\qquad \vdots \\ A_m(x_1 \ \ldots\ x_t) = B_m(x_1 \ldots\ldots x_t) \end{array} \tag{3}$$

where A_i, B_i, $i = 1 \ldots m$ are linear, possibly nonhomogeneous forms in $x_1 \ldots\ldots x_t$, $t \geq n$.

If we wish to optimise the linear form (1) let

$$c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n = z \tag{4}$$

and then the optimum value of z subject to (3) will be the optimum of (1) subject to (3). We then put (4) in the form

$$C_1(x_1x_2 \ldots x_n, z) = C_2(x_1x_2 \ldots x_n, z)$$

where C_1C_2 are also linear forms with $x_1x_2 \ldots x_n$, z on their positive sides. The expression

$$(\mathrm{E}z)\,(\mathrm{E}x_n)\,(\mathrm{E}x_{n-1})\ldots(\mathrm{E}x_1)\,((\bigwedge_{i=1\ldots m}(A_i(x_1\ldots x_t)) = B_i(x_1\ldots x_t))\wedge C_1(x_1\ldots x_n z) = C_2(x_1\ldots x_n)) \tag{5}$$

is then a sentence in Hilbert and Bernays System D and hence we can use Presberger's procedure to eliminate all the variables until only z and constants remain.

At this point I shall explain briefly Presburger's algorithm, or rather an algorithm derived from it. The first step is to change equations into inequalities thus

$$A \leq B \text{ and } A \geq B \text{ instead of } A = B$$

Let us suppose that the equations in (5) are put in the form

$$\begin{array}{ll} a_{i1}x_1 \leq t_i & i \in I_{11} \\ a_{i1}x_1 \geq t_i & i \in I_{21} \end{array}$$

so that $a_{i1} \geq 0$ for all i, I_{11}, I_{21} are index sets, t_i is a function of $x_2 \ldots . x_m$, z, for all i.

Let $a_1 = \text{L.C.M.}\,\{a_{i1} \mid i \in I_{11} \vee I_{21}\}$. Expression (5) is then equivalent to

$$(\mathrm{E}z)\,(\mathrm{E}x_n)\ldots(\mathrm{E}x_1)\begin{cases} a_1x_1 \leq t_i' & i \in I_{11} \\ a_1x_1 \geq t_i' & i \in I_{21}' \\ x_i \geq 0 & \text{for } i = 2, 3, \ldots n\,. \end{cases} \tag{6}$$

where I_{21}' enlarged to include $x_1 \geq 0$, or rather $a_1x_1 \geq 0$ and $t_i' = \frac{a_1}{a_{i1}}, t_i$. If $y = a_1x_1$, (6) is equivalent to

$$(\mathrm{E}z)\,(\mathrm{E}x_n)\ldots(\mathrm{E}x_2)\,(\mathrm{E}y)\begin{cases} y \leq t_i' & i \in I_{11}\,, \\ y \geq t_i' & i \in I_{21}'\,, \\ x_i \geq 0, & i = 2, \ldots n\,. \end{cases} \tag{7}$$

This in turn is equivalent to

$$(\mathrm{E}z)\,(\mathrm{E}x_n)\ldots .(\mathrm{E}x_2)\bigvee_{j\in I_{21}'}\ \bigvee_{r=0,1,\ldots A_1-1}\begin{cases} t_j' + r \leq t_i'\,, & i \in I_{11}\,, \\ t_j' + r \geq t_i' & i \in I_{21} \\ t_j' + r \equiv 0 \bmod a_1 & \\ x_i \geq 0 & i = 2\ldots n\,. \end{cases}$$

So we have a disjunction of sentences each disjunct being a conjunction of inequalities and a congruence, *and* each disjunct containing one variable less than the original expression (6), i.e. we have eliminated x_1 (for proof of this result see [4]). Using the rule

$$(\mathrm{E}x)\,(A_1 \vee A_2 \vee \ldots \vee A_t) \equiv (\mathrm{E}x)\,A_1 \vee (\mathrm{E}x)\,A_2 \vee \ldots \vee (\mathrm{E}x)\,A_t$$

we may consider each disjunct separately. Each disjunct may then be put in the form

$$\begin{cases} a_2x_2 \leq t_i' & i \in I_{12} \\ a_2x_2 \geq t_i' & i \in I_{22}' \\ a_2x_2 \equiv t_0 \bmod a_1 & \end{cases}$$

and subsequently in the form

$$\begin{cases} y \le t_i' & i \in I_{12} \\ y \ge t_i' & i \in I_{22}' \\ y \equiv t_0 \bmod a_1 & \\ y \equiv 0 \bmod a_2 & \\ x_i \ge 0 \text{ for } i = 3 \ldots n & \end{cases}$$

By use of the Chinese Remainder Theorem (see [4]) $y \equiv t_0 \bmod a_1$ and $y \equiv 0 \bmod a_2$ have a simultaneous solution if and only if $t_0 \equiv 0 \bmod (a_1, a_2)$ and if there is a solution $y \equiv kt_0 \bmod [a_1, a_2]$ where k is determined by a_1 and a_2.[1]

We now have a conjunction of inequalities and one congruence. The algorithm is repeated until we are left with a large number of disjuncts each of which is of the form

$$z \equiv q \bmod m \wedge (a < z) \wedge (z < b) \tag{8}$$

Each disjunct may also contain one congruence and some inequalities involving only integers and not z, and it is easily seen, whether these are true or false. If any one is false the entire disjunct is false and may be ignored. If they are all true one may then concentrate on that part of the disjunct in the form of (6). Each of the conjunctions of the form of (8) determine a set of values (possibly empty) for which (5) is true, and the union of these sets is the set of values of z which makes (5) true. Hence we can optimise z, and thus the linear form (1), by choosing the optimum value in this set. In case all the disjunctions (8) are inconsistent, the original equations (2) are inconsistent.

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1. (a_1, a_2) = g.c.d. of a_1 and a_2; $[a_1, a_2]$ = L.C.M. of a_1 and a_2.