248
Notre Dame Journal of Formal Logic
Volume XIII, Number 2, April 1972
NDJFAM

EQUATIONAL CLASSES OF RELATIVE STONE ALGEBRAS

T. HECHT and TIBOR KATRIŇÁK

In this note, we shall show that the lattice of equational classes of relative Stone algebras forms a chain. We shall also show that each class of these algebras can be described by a single equation which joins the equations characterizing Brouwerian algebras. Furthermore, we shall characterize each class of relative Stone algebras in terms of prime filters. In the second part another characterization of the equational classes of L-algebras will be given.

1. *Relative Stone algebras*. A *Brouwerian* algebra is a universal algebra $\langle A, \cup, \cap, *\rangle$, where $\langle A, \cup, \cap\rangle$ is a lattice, and $*$ is a binary operation such that

$$x \leqslant y * z \text{ if and only if } x \cap y \leqslant z \text{ for all elements } x, y, z \in A.$$

Every Brouwerian algebra is distributive and has the greatest element $x * x$, denoted by 1. It is known, that the class of all Brouwerian algebras is an equationally one (see [1, II, §11] or [7] or [8]).

Definition. A relative Stone algebra $\mathfrak{A}$ *is a Brouwerian algebra which satisfies the equation*

$$(x * y) \cup (y * x) = 1 \textit{ for all } x, y \in A.$$

Let F_Θ denote the set $\{x \in A; x \equiv 1(\Theta)\}$ for a congruence relation Θ on a Brouwerian algebra $\mathfrak{A}$. F_Θ forms a filter of $\mathfrak{A}$. The following statement proved in [10] characterizes the congruence relation on a Brouwerian algebra

Lemma 1. *Let* $\mathfrak{A}$ *be a Brouwerian algebra. If* Θ *is a congruence relation on* $\mathfrak{A}$, *then*

$x \equiv y(\Theta)$ *if and only if* $x \cap d = y \cap d$ *for a suitable* $d \in F_\Theta$. *If* F *is a filter of* $\mathfrak{A}$, *then the binary relation* $\Theta(F)$ *defined as follows*:

$x \equiv y(\Theta(F))$ *if and only if* $x \cap d = y \cap d$ *for a suitable* $d \in F$ *is a congruence relation on* $\mathfrak{A}$.

By lemma 1 the lattice of congruence relations $\Theta(A)$ on a Brouwerian

Received November 24, 1970

algebra $\mathfrak{A}$ is isomorphic to the lattice $\mathsf{F}(A)$ of all filters of $\mathfrak{A}$. But $\mathsf{F}(A)$ is distributive. Therefore we can apply to a class of relative Stone algebras the following known result by B. Jónsson [5].

Lemma 2. *Let each algebra* $\mathfrak{A}$ *of an equational class* $\mathcal{K}$ *have a distributive congruence relations lattice. If a finite algebra* $\mathfrak{A} \in \mathcal{K}$ *generates* $\mathcal{K}$ *then each subdirect irreducible algebra* $\mathfrak{B} \in \mathcal{K}$ *is a homomorphic image of a subalgebra of* $\mathfrak{A}$ ($\mathfrak{B} \in \mathsf{HS}(\mathfrak{A})$).

An equational class of algebras is entirely characterized by its subdirectly irreducible algebras. If $\mathcal{K}$ is such a class and $\mathcal{K}_{\mathsf{I}}$ is the subclass of all subdirectly irreducible algebras from $\mathcal{K}$, then $\mathcal{K} = \mathsf{HSP}(\mathcal{K}_{\mathsf{I}})$. It is known that a chain with the greatest element forms a relative Stone algebra, the **RS**-chain algebra. The following lemma (for the proof see [7]) describes the subdirectly irreducible relative Stone algebras.

Lemma 3. *A non-trivial*[1] *relative Stone algebra* $\mathfrak{A}$ *is subdirectly irreducible if and only if* $\mathfrak{A}$ *forms a chain with a dual atom.*[2]

Lemma 4. *A homomorphic image of an* **RS**-*chain algebra is an* **RS**-*chain algebra. If* $\mathfrak{C}$ *is an* **RS**-*chain algebra and* $\mathfrak{A} \subseteq \mathfrak{C}$ *then* $\mathfrak{A}$ *forms a subalgebra of* $\mathfrak{C}$ *if and only if* $1 \in \mathfrak{A}$.

The proof is straightforward.

Let $\mathfrak{C}_n$ denote the **RS**-chain algebra with n elements, $\mathcal{K}_n$ the equational class of all relative Stone algebras generated by $\mathfrak{C}_n$ and $\mathcal{K}_\infty$ the class of all relative Stone algebras. By lemmas 2 and 4 $\mathfrak{C}_k \in \mathcal{K}_n$ if and only if $1 \leqslant k \leqslant n$. Therefore

(1) $\mathcal{K}_1 \subset \mathcal{K}_2 \subset \ldots \subset \mathcal{K}_n \subset \ldots \subset \mathcal{K}_\infty$

Lemma 5. *Every equational class* $\mathcal{K}$ *of relative Stone algebras is generated by its finite subdirect irreducible algebras.*

Proof. Let $\mathfrak{C}$ be an infinite **RS**-chain algebra from $\mathcal{K}$. Let the equation $\mathsf{p}(x_1, \ldots, x_n) = \mathsf{q}(x_1, \ldots, x_n)$ fail to hold in $\mathfrak{C}$ (p, q are terms from the absolutely free relative Stone algebra). Then there exist such elements $a_1, \ldots, a_n \in \mathfrak{C}$ that $\mathsf{p}(a_1, \ldots, a_n) \neq \mathsf{q}(a_1, \ldots, a_n)$. By lemma 4, $\{1, a_1, \ldots, a_n\}$ is a finite subalgebra of $\mathfrak{C}$. Evidently $\{1, a_1, \ldots, a_n\} \in \mathcal{K}$ and $\mathsf{p}(x_1, \ldots, x_n) = \mathsf{q}(x_1, \ldots, x_n)$ does not hold in this finite **RS**-chain algebra.

Theorem 1. *The lattice of all equational classes of relative Stone algebras is isomorphic to the chain* (1) *above of the type* $\omega + 1$.

Proof. Let $\mathcal{K}$ be an equational class of relative Stone algebras. There are two possibilities:

---

1. Containing more than one element.

2. An element c in a lattice with 1 is called a dual atom if $c \leqslant 1$ and $c < x \leqslant 1$ imply that $c = 1$.

(i) There exists a natural number l for which $\mathcal{K}_l \subseteq \mathcal{K}$ and $\mathcal{K}_{l+1} \not\subseteq \mathcal{K}$.
(ii) $\mathcal{K}_n \subseteq \mathcal{K}$ for all natural n.

If $\mathcal{K}_l \subseteq \mathcal{K}$ and $\mathcal{K}_{l+1} \not\subseteq \mathcal{K}$, then $\mathcal{K}$ contains, by lemma 4, only finite **RS**-chain algebras. $\mathfrak{C}_n \in \mathcal{K}$ implies $n \leqslant l$, again by lemma 4. Therefore $\mathcal{K} = \mathcal{K}_l$. If $\mathcal{K}_n \subseteq \mathcal{K}$ for all n, then $\mathcal{K}$ contains all $\mathfrak{C}_n$ and, as a consequence all finite subdirect irreducible relative Stone algebras. Thus $\mathcal{K} = \mathcal{K}_l$ by lemma 5.

Let $\mathrm{p}_n(x_1, \ldots, x_n)$ $(n \geqslant 2)$ denote the polynomial defined recursively:

(i) $\mathrm{p}_2(x_1, x_2) = x_2 \cup (x_2 * x_1)$,
(ii) $\mathrm{p}_n(x_1, \ldots, x_n) = x_n \cup [x_n * \mathrm{p}_{n-1}(x_1, \ldots, x_{n-1})]$.

We can check without difficulties that the **RS**-chain algebra $\mathfrak{C}_n$ $(n \geqslant 2)$, as well as $\mathcal{K}_n$, satisfies the following equations:

(E_n) $\quad (x_1 * x_2) \cup (x_2 * x_3) \cup \ldots \cup (x_n * x_{n+1}) = 1$,
(E'_n) $\quad \mathrm{p}_n(x_1, \ldots, x_n) = 1$.

The **RS**-chain algebra $\mathfrak{C}_m = \{a_1 > a_2 > \ldots > a_n > a_{n+1} \geqslant \ldots \geqslant a_m\}$ $(m > n)$ does not satisfy the equations (E_n) and (E'_n) (set $a_i = x_i$). Thus we have

Theorem 2. *For a Brouwerian algebra* $\mathfrak{A}$, *the following two conditions are equivalent* $(n \geqslant 2)$:

(1) $\mathfrak{A} \in \mathcal{K}_n$,
(2) $\mathfrak{A}$ *satisfies the equation* (E_n) (*or* (E'_n)).

Now we are going to characterize the algebras from $\mathcal{K}_n (n \geqslant 2)$ in terms of prime filters. For the class $\mathcal{K}_\infty$ of all relative Stone algebras it was done in [12].

Theorem 3. *For a Brouwerian algebra* $\mathfrak{A}$, *the following two conditions are equivalent* $(n \geqslant 1)$:

(1) $\mathfrak{A} \in \mathcal{K}_n$,
(2) *The family of all prime filters of* $\mathfrak{A}$ *including a prime filter of* $\mathfrak{A}$ *forms a chain (by inclusion) with at most n elements.*

Proof. Let $\mathfrak{A} \in \mathcal{K}_n (n \geqslant 1)$ and F be a prime filter of $\mathfrak{A}$. Evidently $\mathfrak{A}$ is a relative Stone algebra. The quotient algebra $\mathfrak{A}/F$ is a chain (see [4, lemma 1.1]). Since $\mathfrak{A}/F \in \mathcal{K}_n$, $\mathfrak{A}/F$ has at most n elements. Let $\nu\colon \mathfrak{A} \to \mathfrak{A}/F$ be the natural epimorphism. For a prime filter $F' \supseteq F$ the image $\nu(F')$ is a filter of $\mathfrak{A}/F$ and there exists such an element $f' \in \mathfrak{A}/F$ that $\nu(F') = [f')$[3] holds. By lemma 1 and $F' \supseteq F$ it is clear that the inverse image $\nu^{-1}\{[f')\} = F'$. Let now F_1, F_2 be prime filters of $\mathfrak{A}$ containing F. There exist elements $f_1, f_2 \in \mathfrak{A}/F$ with $\nu(F_1) = [f_1)$, $\nu(F_2) = [f_2)$. $\mathfrak{A}/F$ is a chain and therefore $F_1 = \nu^{-1}\{[f_1)\} \subseteq F_2 = \nu^{-1}\{[f_2)\}$ or $F_2 \subseteq F_1$. For $F_1 \neq F_2$ $[f_1) \neq [f_2)$ must hold. Thus we have proved the condition (2).

Let the condition (2) be fulfilled by a Brouwerian algebra $\mathfrak{A}$. By [12], $\mathfrak{A}$

3. $[f') = \{x \in \mathfrak{A}/F;\, x \geqslant f'\}$.

is a relative Stone algebra. Let $\mathfrak{I}$ denote the set of all prime filters of $\mathfrak{A}$. $\mathfrak{A}$ is a distributive lattice with 1 and by the known "Stone's theorem" $[1) = \bigwedge_{F \in \mathfrak{I}} F$ holds. By lemma 1, $\bigwedge_{F \in \mathfrak{I}} \Theta(F)$ is the identical congruence relation on $\mathfrak{A}$. Therefore $\mathfrak{A}$ is a subdirect product of relative Stone algebras $\mathfrak{A}/\Theta(F) = \mathfrak{A}/F$ $(F \in \mathfrak{I})$. Above we have proved that $f' \in \mathfrak{A}/F$ $(F \in \mathfrak{I})$ if and only if $\nu^{-1}\{[f')\}$ is a prime filter containing F. By our assumption, $\mathfrak{A}/F$ $(F \in \mathfrak{I})$ is a chain with at most n elements which implies $\mathfrak{A}/F \in \mathscr{K}_n$ for all $F \in \mathfrak{I}$ and consequently $\mathfrak{A} \in \mathscr{K}_n$.

2. L-*algebras* An universal algebra $\langle A, \cup, \cap, *, 0\rangle$ is called a *Heyting* algebra if $\langle A, \cup, \cap, *\rangle$ is a Brouwerian algebra and 0 is the smallest element of A. An L-algebra is a Heyting algebra for which $\langle A, \cup, \cap, *\rangle$ is a relative Stone algebra (see [4]). It is easy to see that both classes of Heyting and L-algebras are equational ones. All the results of section 1 are true also for the L-algebras. Especially, a chain C with the greatest element and the smallest element is an L-algebra, the chain L-algebra. An L-algebra $\mathfrak{A}$ is subdirect irreducible if and only if $\mathfrak{A}$ forms a chain L-algebra with a dual atom. Let $\mathscr{L}_n$ denote the equational class of all L-algebras generated by the chain L-algebra $\mathfrak{C}_n$ with n elements. Let $\mathscr{L}_\infty$ denote the class of all L-algebras. Repeating the proof of Theorem 1 we can obtain an analogous result for L-algebras: *The lattice of all equational classes of* L*-algebras forms a chain*

(2) $\mathscr{L}_1 \subset \mathscr{L}_2 \subset \ldots \subset \mathscr{L}_n \subset \ldots \subset \mathscr{L}_\infty$

of type $\omega + 1$. Furthermore for a Heyting algebra $\mathfrak{A}$: $\mathfrak{A} \in \mathscr{L}_n$ $(n \geq 2)$ *if and only if* $\mathfrak{A}$ *satisfies the equation* (E_n) (*or* $(\mathrm{E}_n'))$.

For our next considerations we need some new conceptions concerning Brouwerian, Heyting and L-algebras. A Brouwerian algebra satisfies the following equations (see [1])

(3) $(x \cap y) * z = x * (y * z)$,

(4) $x * (y \cap z) = (x * y) \cap (x * z)$.

In a Heyting algebra the elements $a * 0$ will be denoted by a^*. An element a of a Heyting algebra $\mathfrak{A}$ will be called *closed* (dense) if $a = a^{**}$ $(a^* = 0)$. The set $\mathrm{C}(A) = \{a \in A;\ a = a^{**}\}$ forms a Boolean algebra, the Boolean algebra of closed elements. The set $\mathrm{D}(A) = \{a \in A;\ a^* = 0\}$ forms a filter, the filter of dense elements. It is easy to see that $\mathrm{D}(A)$ forms a Brouwerian algebra (more precisely the Brouwerian subalgebra of $\mathfrak{A}$). For each element x of a Heyting algebra $\mathfrak{A}$ there exists a suitable dense element $d \in \mathrm{D}(A)$ for which

(5) $x = x^{**} \cap d$

holds.

It is known that $\mathrm{C}(A)$ is a sublattice of $\mathfrak{A}$ if $\mathfrak{A}$ is an L-algebra. Moreover, for a closed element y of an L-algebra the element $x * y$ is also closed and

(6) $x * y = x^* \cup y$

is true (see [10, lemma 4.2]). In [2], [6] and [11], the L-algebras were characterized as follows:

Theorem 4. *A Heyting algebra* $\mathfrak{A}$ *is an* L*-algebra if and only if the following conditions are fulfilled*:

(i) $\mathrm{C}(A)$ *is a subalgebra of* $\mathfrak{A}$,
(ii) $\mathrm{D}(A)$ *is a relative Stone algebra.*

We wish to give an analogous characterization of L-algebras from $\mathscr{L}_n$ $(n \geqslant 2)$.

Lemma 6. *Let* $\mathscr{H}$ *be an equational class of Heyting algebras. Then the class of all Brouwerian algebras* $\mathrm{D}(A)$ *for* $\mathfrak{A} \in \mathscr{H}$ *is an equational one.*

Proof. Let $\mathscr{B}$ denote the class of all algebras $\mathrm{D}(A)$ for $\mathfrak{A} \in \mathscr{H}$. It satisfies to prove that direct products, epimorphic images and subalgebras of members from $\mathscr{B}$ are again algebras from $\mathscr{B}$. Let $\mathrm{D}(A_\alpha) \in \mathscr{B}$ for $\alpha \in \mathrm{I}$ and $A = \prod A_\alpha$. Evidently $\mathfrak{A} \in \mathscr{H}$. Then $a = (a_\alpha)_{\alpha \in \mathrm{I}} \in \mathrm{D}(A)$ if and only if $a_\alpha \in \mathrm{D}(A_\alpha)$ for all $\alpha \in \mathrm{I}$. Thus $\mathrm{D}(A) = \prod \mathrm{D}(A_\alpha) \in \mathscr{B}$.

Let now $\mathrm{D}(A_2) \in \mathscr{B}$ be an epimorphic image of $\mathrm{D}(A_1) \in \mathscr{B}$. By lemma 1, there exists such a filter F of $\mathrm{D}(A_1)$ that $\mathrm{D}(A_1)/F$ is isomorphic to $\mathrm{D}(A_2)$. Consider $A_1/F \in \mathscr{H}$. $x \equiv y(\Theta(F))$ implies $x^{**} \equiv y^{**}(\Theta(F))$ and consequently $x^{**} \cap d = y^{**} \cap d$ for a suitable $d \in F$. Since $F \subseteq \mathrm{D}(A_1)$, we obtain $x^{**} = y^{**}$. Denote $x \to \bar{x}$ the natural epimorphism from A_1 onto A_1/F. If $\bar{x} \in \mathrm{D}(A_1/F)$, then $x^{**} \in F$ for $x \in \bar{x}$ and because $F \subseteq \mathrm{D}(A_1)$, we have $x^{**} = 1$. Hence $x \in \mathrm{D}(A_1)$. Evidently $x \in \mathrm{D}(A_1)$ implies $\bar{x} \in \mathrm{D}(A_1/F)$. Therefore $\mathrm{D}(A_1/F)$ is isomorphic to $\mathrm{D}(A_1)/F$.

Let, finally, $\mathfrak{G}$ be a Brouwerian subalgebra of $\mathrm{D}(A) \in \mathscr{B}$. $\mathfrak{G}' = \mathfrak{G} \cup \{0\}$ is a Heyting subalgebra of $\mathfrak{A} \in \mathscr{H}$. Hence $\mathfrak{G}' \in \mathscr{H}$. It is easy to see that $\mathrm{D}(G') = \mathfrak{G}$ and, consequently, $\mathfrak{G} \in \mathscr{B}$.

Lemma 7. $\mathscr{K}_{n-1}$ *is the class of all* $\mathrm{D}(A)$ *for* $\mathfrak{A} \in \mathscr{L}_n$ $(n \geqslant 2)$.

Proof. Let $\mathscr{K}$ denote the class of all algebras $\mathrm{D}(A)$ for $\mathfrak{A} \in \mathscr{L}_n$ $(n \geqslant 2)$. By lemma 6, $\mathscr{K}$ is an equational class of relative Stone algebras. $\mathscr{L}_n$ is generated by the chain L-algebra $\mathfrak{C}_n$. $\mathrm{D}(C_n) = \mathfrak{C}_{n-1}$ and $\mathfrak{C}_{n-1}$, considered as an RS-chain algebra, generates $\mathscr{K}_{n-1}$. Thus $\mathscr{K}_{n-1} \subseteq \mathscr{K}$. If $\mathfrak{C}_n = \mathrm{D}(A)$ for an algebra $\mathfrak{A} \in \mathscr{L}_n$, then $\mathfrak{C}_{n+1} = \mathfrak{C}_n \cup \{0\}$ would be an L-subalgebra of $\mathfrak{A}$ and, thus, $\mathfrak{C}_{n+1} \in \mathscr{L}_n$ which would be a contradiction. Hence the subdirect irreducible algebras from $\mathscr{K}$ are $\mathfrak{C}_l$ for $1 \leqslant l \leqslant n - 1$ and this implies $\mathscr{K}_{n-1} = \mathscr{K}$.

Theorem 5. *Let* $\mathfrak{A}$ *be a Heyting algebra. Then* $\mathfrak{A} \in \mathscr{L}_n$ $(n \geqslant 2)$ *if and only if the following conditions are fulfilled*:

(i) $\mathrm{C}(A)$ *is a subalgebra of* A.
(ii) $\mathrm{D}(A) \in \mathscr{K}_{n-1}$.

Proof. Since $\mathscr{L}_n \subset \mathscr{L}_\infty$, the necessity follows by Theorem 4 and lemma 7. Now we can assume that $n \geq 2$ and a Heyting algebra $\mathfrak{A}$ satisfies the conditions (i) and (ii). Evidently $\mathfrak{A} \in \mathscr{L}_\infty$, by Theorem 4. We shall prove the algebra $\mathfrak{A}$ satisfies the equation (E_n). For the elements $x_1, \ldots, x_{n+1} \in A$ there exist such dense elements $d_1, \ldots, d_{n+1} \in \mathrm{D}(A)$ that

$$x_i = x_i^{**} \cap d_i \quad (1 \leq i \leq n+1).$$

Now $x_i * x_{i+1} = (x_i^{**} \cap d_i) * (x_{i+1}^{**} \cap d_{i+1}) = [(x_i^{**} \cap d_i) * x_{i+1}^{**}] \cap [(x_i^{**} \cap d_i) * d_{i+1}] = [d_i * (x_i^{**} * x_{i+1}^{**})] \cap [x_i^{**} * (d_i * d_{i+1})]$ for $1 \leq i \leq n$, by (4) and (5). Thus,

$(x_1 * x_2) \cup (x_2 * x_3) \cup \ldots \cup (x_n * x_{n+1}) = [(x_1^{**} \cap d_1) * (x_2^{**} \cap d_2)] \cup \ldots \cup [(x_n^{**} \cup d_n) * (x_{n+1}^{**} \cap d_{n+1})] = [(d_1 * (x_1^{**} * x_2^{**})) \cap (x_1^{**} * (d_1 * d_2))] \cup \ldots \cup [(d_n * (x_n^{**} * x_{n+1}^{**})) \cap (x_n^{**} * (d_n * d_{n+1}))]$.

By (6), we have $d_i * (x_i^{**} * x_{i+1}^{**}) = d_i^* \cup (x_i^{**} * x_{i+1}^{**}) = x_i^* \cup x_{i+1}^{**}$ for all $1 \leq i \leq n$. If we put

$$a_i = x_i^* \cup x_{i+1}^{**} \text{ and } b_i = x_i^{**} * (d_i * d_{i+1})$$

for $1 \leq i \leq n$, then we obtain

(iii) $(x_1 * x_2) \cup \ldots \cup (x_n * x_{n+1}) = (a_1 \cap b_1) \cup \ldots \cup (a_n \cap b_n)$.

Since $b_i \geq d_i * d_{i+1} \in \mathrm{D}(A)$ $(1 \leq i \leq n)$ and $\mathrm{D}(A) \in \mathscr{K}_{n-1}$, we can conclude by Theorem 2 and (ii)

(iv) $b_1 \cup \ldots \cup b_n \geq (d_1 * d_2) \cup \ldots \cup (d_{n-1} * d_n) = 1$ and
$b_1 \cup \ldots \cup b_{n-1} \cup a_n \geq (d_1 * d_2) \cup \ldots \cup (d_{n-1} * d_n) = 1$.

Evidently $a_i \cup a_{i+1} = (x_i^* \cup x_{i+1}^{**}) \cup (x_{i+1}^* \cup x_{i+2}^{**}) = 1$ $(1 \leq i \leq n-1)$ because $x_{i+1}^* \cup x_{i+1}^{**} = 1$ follows from (i). Hence

(v) $c_1 \cup \ldots \cup c_n = 1$, if $c_i = a_i$, $c_{i+1} = a_{i+1}$ for some $1 \leq i \leq n-1$ and $c_j \in \{a_j, b_j\}$ for $1 \leq j \leq n$ and $i \neq j \neq i+1$.

Since $d_i * d_{i+1} \geq 0$ $(1 \leq i \leq n)$, we obtain

$$x_i^* = x_i^{**} * 0 \leq x_i^{**} * (d_i * d_{i+1}) = b_i$$

and, as a cconsequence,

(vi) $a_{i-1} \cup b_i \geq (x_{i-1}^* \cup x_i^{**}) \cup x_i^* \geq x_i^{**} \cup x_i^* = 1$ for $2 \leq i \leq n$.

Therefore,

(vii) $c_1 \cup \ldots \cup c_n = 1$, if $c_i = a_i$, $c_{i+1} = b_{i+1}$ for some $1 \leq i \leq n-1$ and $c_j \in \{a_j, b_j\}$ for $1 \leq j \leq n$ and $i \neq j \neq i+1$.

It is known that a Heyting algebra is distributive. Applying the distributivity law on the right hand side of (iii) and considering (iv)-(vii), we obtain,

$$(x_1 * x_2) \cup (x_2 * x_3) \cup \ldots \cup (x_n * x_{n+1}) = 1,$$

what was to be proved.

REFERENCES

[1] Birkhoff, G., *Lattice Theory*, American Mathematical Society, Colloquium Publications, vol. 25, third edition (1967).

[2] Chen, C. C., and G. Grätzer, "Stone lattices II. Structure theorems," *Canadian Journal of Mathematics*, vol. 21 (1969), 895-903.

[3] Grätzer, G., and E. T. Schmidt, "On a problem of M. H. Stone," *Acta Mathematica Academiae Scientiarum Hungariae*, vol. 8 (1957), pp. 455-460.

[4] Horn, A., "Logic with truth values in a linearly ordered Heyting algebra," *The Journal of Symbolic Logic*, vol. 34 (1969), pp. 395-408.

[5] Jónsson, B., "Algebras whose congruence lattices are distributive," *Mathematica Scandinavia*, vol. 21 (1967), pp. 110-121.

[6] Katriňák, T., "Die Kennzeichnung der distributiven pseudokomplementären Halbverbände," *Journal für die Reine und angerwandte Mathematik*, vol. 241 (1970), pp. 160-179.

[7] Katriňák, T., "Remarks on the W. C. Nemitz's paper 'Semi-Boolean lattices,'" *Notre Dame Journal of Formal Logic*, vol. XI (1970), pp. 425-430.

[8] Katriňák, T., and A. Mitschke, "Postalgebren und Stonesche Verbände der Ordnung *n*," *Colloquium Mathematicae* (to appear).

[9] Lee, K. B., "Equational classes of distributive pseudo-complemented lattices," (Preprint).

[10] Nemitz, W. C., "Implicative semi-lattices," *Transactions of the American Mathematical Society*, vol. 117 (1965), pp. 128-142.

[11] Nemitz, W. C., "Semi-Boolean lattices," *Notre Dame Journal of Formal Logic*, vol. X (1969), pp. 235-238.

[12] Varlet, J., "On the characterization of Stone lattices," *Acta Scientiarum Mathematicarum*, vol. 27 (1966), pp. 81-84.

Katedra Algebry a Teórie Čisel
Prirodovedeckej Faculty
Univerzita Kamenskeho
Bratislava, Czechoslovakia