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REAL FIELDS WITH CHARACTERIZATION OF THE NATURAL NUMBERS

EUGENE W. MADISON

Introduction. This paper is a sequel to [3]. Its purpose is two-fold: On the one hand, it is to inform the reader of the state of affairs regarding the concept of structures being elementarily closed (relative to the natural numbers) since the publication of Theorems 3D, 4A of [2] where the concept (without name) was introduced for the first time. And, on the other hand, our purpose is to give some clarification to results of [3]. In [3] the term "elementarily closed" was introduced to apply to the general structure which satisfies the conclusions of Theorems 3D, 4A of [2]. However, recent studies have led us to conclude that a stronger form of our definition is more interesting and natural from the point of view of our results in [2]. We shall attempt to be more specific after giving a precise frame of reference.

1 *Basic Definitions and Remarks.* We are here interested in structures of the form $\mathcal{J} = \{F, \mathcal{N}_0, +, \cdot, \leq, 0\}$, where F is the set-part for a field of characteristic zero, $\mathcal{N}_0$ the set of natural numbers, "+" and "·" the ternary relations of addition and multiplication, respectively, and "≤" the binary relation of order. (In some cases we choose to drop "<".) By the language of $\mathcal{J}$, say $\Sigma_{\mathcal{J}}$, is meant a convenient formulation of the lower predicate calculus which contains the extralogical constants $\mathsf{N}(x)$, $\mathsf{E}(x, y)$, $\mathsf{S}(x, y, z)$, $\mathsf{P}(x, y, z)$ and $\mathsf{H}(x, y)$ whose intended interpretations are "$x \in \mathcal{N}_0$", $x = y$, $x + y = z$, $x \cdot y = z$, and $x < y$, respectively.

Let $\mathcal{J}$ and $\mathcal{G}$ be two structures and $A \subseteq F \cup G$. Recall that $\mathcal{J}$ and $\mathcal{G}$ are said to be *elementarily equivalent* with respect to A in case $\mathcal{J} \vDash X$ if and only if $\mathcal{G} \vDash X$, for any sentence X which is defined in the language of $\mathcal{J}$ (and $\mathcal{G}$) and whose individual constants correspond to elements of A. $\mathcal{J}$ and $\mathcal{G}$ are said to be *elementarily equivalent* in case they are elementarily equivalent with respect to ϕ. $\mathcal{J}$ is an *elementary extension* of $\mathcal{G}$ in case $\mathcal{J}$ is an extension of $\mathcal{G}$ and $\mathcal{J}$ and $\mathcal{G}$ are elementarily equivalent with respect to G.

It is interesting to observe that a certain abnormality occurs for

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structures viewed within our framework where we have chosen to distinguish the natural numbers. For example, the field $\mathcal{R}$ of real numbers is not an elementary extension of the field $\mathcal{R}_0$ of real algebraic numbers. See [2] and [5]. This is to say, the theorem of Tarski, namely, "a real-closed field is an elementary extension of each of its real-closed subfields" does not hold.

We now give the stronger form of the definition of "elementarily closed" which was alluded to in the introduction. The same term (elementarily closed) has been retained for the present concept.

Definition. A structure $\mathcal{I}$ (abbreviated $\mathcal{I} = \{F; \mathcal{N}_0\}$) is *elementarily closed* (relative to $\mathcal{N}_0$) in case

(a) every proper elementary extension of $\mathcal{I}$ enlarges $\mathcal{N}_0$, and

(b) whenever extensions of $\mathcal{I}$, $\mathcal{I}^* = \{F^*; \mathcal{N}^*\}$ and $^*\mathcal{I} = \{^*F; \mathcal{N}^*\}$ are elementarily equivalent with respect to $F \cup \mathcal{N}^*$ then $\mathcal{I}^* \cong {}^*\mathcal{I}$.

It is clear that there are lots of structures which are not elementarily closed relative to $\mathcal{N}_0$. Let $\mathcal{I} = \{F, \mathcal{N}_0\}$ be any uncountable structure (having only finitely many relations). A theorem of Tarski and Vaught (see [6]) guarantees the existence of a proper elementary substructure $\mathcal{I}' = \{F', \mathcal{N}_0'\}$ of $\mathcal{I}$. Clearly, $\mathcal{N}_0' = \mathcal{N}_0$; therefore, $\mathcal{I}'$ is not elementarily closed relative to $\mathcal{N}_0$.

The main theorem of [3] gives a sufficient condition for a structure to be elementarily closed relative to $\mathcal{N}_0$. However, our application of this theorem (i.e., the last theorem of [3]) which asserts, "*every* AD-*structure* $\mathcal{I} = \{F, \mathcal{N}_0\}$ *is elementarily closed relative to* $\mathcal{N}_0$", is false. A counterexample to this theorem and an alternative theorem (with stronger hypotheses) are discussed in the sections that follow.

2 *A Counterexample to a Theorem of* [3]. Let $\{C; \mathcal{N}_0\}$ denote the field of complex numbers with the natural numbers $\mathcal{N}_0$ distinguished. Let $\mathcal{R}^C$ denote the field of computable real numbers. It is easily shown that the transcendence degree of $\mathcal{R}^C$ over Q is $\aleph_0$. Simply, let $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\}$ be a denumerable set of real algebraic numbers which is linearly independent over Q. The set $\{e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n}, \ldots\} \subseteq \mathcal{R}^C$ and is algebraically independent over Q. This is an immediate consequence of a theorem of Lindemann which asserts that if $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is a linearly independent set of algebraic numbers over Q then $\{e^{\beta_1}, e^{\beta_2}, \ldots, e^{\beta_n}\}$ is algebraically independent over Q.

Now, let $\{t_1, t_2, \ldots, t_n, \ldots\}$ be a transcendence basis of $\mathcal{R}^C$ over Q. Then, clearly, the algebraic closure of $Q(t_1, \ldots, t_n, \ldots)$ is $\mathcal{R}^C(i)$.

Theorem (A. Robinson). *If A is an algebraically closed field of characteristic zero, $\{t_1, \ldots, t_n, \ldots\}$ a subset of A algebraically independent over Q, and $\overline{A}$ the algebraic closure of $Q(t_1, \ldots, t_n, \ldots)$, then A is an elementary extension of $\overline{A}$.*

An immediate conclusion is that $\{C, \mathcal{N}_0\}$ is an elementary extension of

$\{\mathcal{R}^C(i), \mathcal{N}_0\}$. In other words, $\mathcal{R}^C(i)$ is not elementarily closed relative to $\mathcal{N}_0$. In [1] we proved that $\mathcal{R}^C$ is an AD-structure; hence, $\mathcal{R}^C(i)$ is an AD-structure.

Our conclusion is that not every AD-structure is elementarily closed relative to $\mathcal{N}_0$.

3 *Structures Elementarily Closed Relative to $\mathcal{N}_0$.*

Lemma 1. *Let $\{\mathcal{R}_0, \mathcal{N}_0\}$ be a structure such that there is an injective function ψ: $\mathcal{R}_0 \to \mathcal{N}_0$ such that the predicate $\psi(x) = y$ is expressible in $\Sigma_{\mathcal{R}_0}$. Then $\mathcal{R}_0$ is elementarily closed relative to $\mathcal{N}_0$.*

Proof. Let ϕ: $\mathcal{N}_0 \to \mathcal{R}_0$ be defined by:

$$\phi(n) = \begin{cases} \alpha, \text{ in case } \psi(\alpha) = n \\ 0, \text{ otherwise.} \end{cases}$$

Clearly, the predicate $\phi(x) = y$ is expressible in $\Sigma_{\mathcal{R}_0}$. Also, ϕ is N-bijective in the sense of [3]. Therefore, by our theorem of [3], $\mathcal{R}_0$ is elementarily closed relative to $\mathcal{N}_0$.

Lemma 2. *If $\mathcal{R}_0$ is an* AD-*structure such that some admissible indexing of $\mathcal{R}_0$ is expressible in $\Sigma_{\mathcal{R}_0}$ then $\mathcal{R}_0$ is elementarily closed relative to $\mathcal{N}_0$.*

The following theorem shows that the last theorem of [3] does hold for a reasonably large class of AD-structures.

Theorem 3. *If $\{\mathcal{R}_0, \mathcal{N}_0\}$ is an* AD-*ordered subfield of real numbers, then $\mathcal{R}_0$ is elementarily closed relative to $\mathcal{N}_0$.*

Proof. First of all, we observe that the greatest integer function "$[x] = y$" is expressible in $\Sigma_{\mathcal{R}_0}$. Secondly, if λ is a fixed admissible indexing of $\mathcal{R}_0$ then $\lambda | \mathcal{N}_0$ is expressible in $\Sigma_{\mathcal{R}_0}$. Now we define ψ: $\mathcal{R}_0 \to \mathcal{N}_0$ by: $\psi(\alpha) = \beta$ if and only if

3.1. $\mathsf{K}(\beta) \wedge (\forall n)[\mathsf{N}(n) \Rightarrow (\exists t)[([(n+1)\alpha] = t) \wedge (\bar{[}\lambda(n+1) \cdot{}' \beta\bar{]} = \lambda(t))]]$,

holds in $\mathcal{R}_0$, where "$[z]$" denotes the greatest integer function, "$\bar{[}z\bar{]}$" denotes the greatest integer function in the arithmetical representation of $\mathcal{R}_0$, "$\cdot'$" denotes multiplication in the arithmetical representation of $\mathcal{R}_0$, and $\mathsf{K}(z)$ denotes the predicate "$z \in \lambda(\mathcal{R}_0)$."

Clearly, 3.1 is expressible in $\Sigma_{\mathcal{R}_0}$. Since ψ is an isomorphism between $\mathcal{R}_0$ and its arithmetical representation, it is clear that ψ is injective. Therefore, by Lemma 1, $\mathcal{R}_0$ is elementarily closed relative to $\mathcal{N}_0$.

Remarks. We observe in [3] that several familiar structures are elementarily closed relative to $\mathcal{N}_0$, for example, the ring of Gaussian integers, the field of real algebraic numbers, the fields of solvable and constructible numbers.

In earlier paragraphs we showed that $\mathcal{R}^C(i)$ is not elementarily closed relative to $\mathcal{N}_0$. However, as an immediate consequence of Theorem 3, we have that $\mathcal{R}^C$ is elementarily closed relative to $\mathcal{N}_0$.

It is interesting to raise the question as to whether or not the converse of Theorem 3 holds. A consequence of our next theorem is that the converse of Theorem 3 is false.

Theorem 4. *There exists a subfield* $\mathcal{R}_1$ *of real numbers such that* $\mathcal{R}_1$ *is elementarily closed (relative to* $\mathcal{N}_0$*) and* $\mathcal{R}_1$ *is not* AD*-ordered.*

Proof. Let $\mathrm{B}(x, y)$ be the predicate of 3.1. Note that $\mathrm{B}(x, y)$ defines a function ψ: $\mathcal{R}^C \to \mathcal{N}_0$ (i.e., $\psi(a) = m$ if and only if $\mathrm{B}(\alpha, m)$ holds in $\mathcal{R}^C$) such that ψ is injective.

Let $\mathcal{R}'$ be any subfield of $\mathcal{R}^C$. Then the function φ: $\mathcal{R}' \to \mathcal{N}_0$, defined by $\varphi(\alpha) = m$ if and only if $\mathrm{B}(\alpha, m)$ holds in $\mathcal{R}^C$ (i.e., $\varphi = \psi|_{\mathcal{R}'}$), is a function of the type of Lemma 1, since $(\forall y)(\exists! x)\mathrm{B}(x, y)$ holds in every subfield of $\mathcal{R}^C$. It follows from Lemma 1 that $\mathcal{R}'$ is elementarily closed (relative to $\mathcal{N}_0$). In summary, every subfield of $\mathcal{R}^C$ is elementarily closed (relative to $\mathcal{N}_0$).

Since there are only countably many arithmetically definable ordered fields, it suffices to show that $\mathcal{R}^C$ has uncountably many non-(order) isomorphic subfields. Indeed, we have established that the transcendence degree of $\mathcal{R}^C$ over Q is $\aleph_0$, hence any such field has infinitely many non-(order) isomorphic subfields.

Let $T = \{S | S \subseteq \Delta\}$, where Δ is a transcendence basis for $\mathcal{R}^C$ over Q. Clearly, T is uncountable. Let $\mathrm{K}_S = Q(S)$. If $S \neq S'$ then clearly K_S is not order isomorphic to $\mathrm{K}_{S'}$, lest they be identical, which is impossible. But, since $S \neq S'$ there exists a $t \in S'$ such that $t \notin S$ (or just the other way). For notation, assume the former. Now $S \cup \{t\}$ is algebraically independent; therefore, $t \notin \mathrm{K}_S$, while $t \in \mathrm{K}_{S}$. Our theorem is proved.

4 *Some Embedding Theorems.* The following is well-known.

(c) *A field is Archimedean ordered if and only if it can be embedded in the field of real numbers.*

In this section we seek a non-standard analogue of (c). Let $\mathcal{R} = \{R, \mathcal{N}_0, +, \cdot, \leq\}$ be the ordered field of real numbers with $\mathcal{N}_0$ the natural numbers as a distinguished subject. Any proper elementary extension of $\mathcal{R}$ will be referred to as a *non-standard model of analysis*. It is easy to see that any non-standard model of analysis enlarges the cardinality of $\mathcal{N}_0$.

Before stating our analogue of (c) we prove the following lemma.

Lemma 5. *If* $\mathcal{R}_2 = \{R_2; \mathcal{N}_0\}$ *is an extension of* $\mathcal{R}^C$*, then* $\mathcal{R}^C$ *is arithmetically definable in* $\mathcal{R}_2$ *(i.e., there is a predicate* $\mathrm{G}(x)$ *which is expressible in the language of* $\mathcal{R}_2$ *such that* $\mathrm{G}(\alpha)$ *holds in* $\mathcal{R}_2$ *if and only if* $\alpha \in R^C$*).*

Proof. Our method of proof is reminiscent of the proof that $\mathcal{R}^C$ is an AD-ordered field. See [1]. Since the set of partial recursive functions is recursively enumerable, we can find an arithmetical predicate, say $\mathrm{C}(x, y, z)$, (i.e., $\mathrm{C}(x, y, z)$ is expressible in the language of $\mathcal{R}_2$) which expresses "a total function ψ_x is defined and has value z at y." Of course, ψ_x: $\mathcal{N}_0 \to \mathcal{N}_0$. Let $\mathrm{G}(u)$ be the predicate

4.0 $(\exists x)[\mathrm{N}(x) \wedge (\forall y)(\forall z)[\mathrm{N}(y) \wedge \mathrm{N}(z) \wedge \mathrm{C}(x, y, z) \wedge (y \neq 0)] \Rightarrow [|u - \rho(z)| < y^{-1}]]$, where ρ is a fixed effective indexing of natural numbers with rationals. Clearly, $\mathsf{G}(u)$ is expressible in the language of $\mathcal{R}_2$ and $\mathsf{G}(\alpha)$ holds in $\mathcal{R}_2$ if and only if $\alpha \in R^C$. Our lemma is proved.

We may, in fact, take as our definition of $\mathcal{R}^C$ the following: $R^C = \{\alpha \in R \mid \mathsf{G}(\alpha)$ holds in $R\}$. *An immediate consequence of Lemma 5 is that* every element of R^C is definable in the sense that if $a \in R^C$ then there is a predicate $\mathsf{F}_a(x)$ which is defined in $\mathcal{N}$ such that (i) $\mathsf{F}_a(a)$ holds in $\mathcal{R}^C$ and (ii) $(\forall y)\,[\mathsf{F}_2(y) \Rightarrow \mathsf{E}(a, y)]$ holds in $\mathcal{R}^C$. Indeed, each $a \in \mathcal{R}^C$ is definable in every subfield of $\mathcal{R}^C$ containing a. Simply use 4.0.

Now we proceed with our discussion of a non-standard analogue of property (c). Consider the following:

(d) Let $\mathcal{R}^* = \{R^*; \mathcal{N}^*\}$ be a non-standard model of analysis. If $^*\mathcal{R} = \{^*R; \mathcal{N}^*\}$ is a model of $K \cup P$, where K is the axioms for the concept of "Archimedean ordered field" and P the true sentences of $\mathcal{N}_0$ relativised by $\mathsf{N}(x)$, then $^*\mathcal{R}$ can be embedded in $\mathcal{R}^*$.

The statement of (d) is a direct analogue of (c) and can be shown to be false. Simply let $\mathsf{X}_{r,s}(\beta)$ denote the sentence $r < \beta < x$, where $r \in Q$, $s \in (Q^* - Q)$ (where Q^* denotes the quotient field of $\mathcal{N}^*$ in $\mathcal{R}^*$), and β an individual constant. Let $U = \{\mathsf{X}_{r,s}(\beta) \mid r \in Q \mid \wedge\, s \in (Q^* - Q)\}$. We claim that the set V, consisting of the complete diagram of $\mathcal{N}^*$ union with U and the field axioms, is consistent. Easily, any finite subset of V is consistent.

Let $\mathcal{R} = \{R; \mathcal{N}^{**}\}$ be a model of V. Clearly, $\mathcal{N}^{**} \supseteq \mathcal{N}^*$ and, moreover, $\mathcal{R}_1 = \{Q^*(\beta), \mathcal{N}^*\}$ can be shown to be a model of $K \cup P$. Clearly, $\mathcal{R}_1$ cannot be embedded in $\mathcal{R}^*$ since no element of $\mathcal{R}^*$ determines the same "rational" cut as β determines in $\mathcal{R}_1$.

Our next theorem and its converse constitute what seems to be about the strongest non-standard analogue of property (a) that one can expect.

Theorem 6. *For each model of analysis* $\mathcal{R}^* = \{R^*; \mathcal{N}^*\}$ *there is an elementary extension of* $\mathcal{R}^C$, *say* $^*\mathcal{R} = \{R; \mathcal{N}^*\}$, *(i.e., corresponding to the same model of arithmetic* $\mathcal{N}^*$*) such that* $\mathcal{R}^*$ *contains an isomorphic copy of* $^*\mathcal{R}$.

Proof. Let $\mathsf{G}(u)$ be as in Lemma 5. Let $^*R^C = \{\alpha \in R^* \mid \mathsf{G}(\alpha)$ holds in $\mathcal{R}^*\}$. Clearly, $^*\mathcal{R}^C$ is an extension of $\mathcal{R}^C$. Let $(\exists z)\mathsf{X}(z)$ be such that it is defined in $\mathcal{R}^C$ and holds in $^*\mathcal{R}^C$. Then $(\exists z)[\mathsf{X}(z) \wedge \mathsf{G}(z)]$ holds in $\mathcal{R}^*$, whence holds in $\mathcal{R}$. Therefore, $[\mathsf{X}(a) \wedge \mathsf{G}(a)]$ holds in $\mathcal{R}$, for some $a \in R$, i.e., $\mathsf{X}(a)$ holds in $\mathcal{R}$, $a \in R^C$. Obviously, $\mathsf{X}(a)$ holds in $^*\mathcal{R}^C$, $a \in \mathcal{R}^C$. Therefore, by the classical theorem of Tarski and Vaught, $^*\mathcal{R}^C$ is an elementary extension of $\mathcal{R}^C$.

Theorem 7. *Let* $\mathcal{R}_1$ *be a computable ordered subfield of* $\mathcal{R}$. *Any model of analysis* $\mathcal{R}^* = \{R^*; \mathcal{N}^*\}$ *contains an elementary extension of* $\mathcal{R}_1$, *say* $^*\mathcal{R} = \{^*R; \mathcal{N}^*\}$ *(i.e., corresponding to the same* $\mathcal{N}^*$*).*

Proof. From the proof of Theorem 6 it is clear that it suffices to show that $\mathcal{R}_1$ is arithmetically definable in $\mathcal{R}^C$ in the sense of Lemma 5. Let $\mathsf{A}(x, u)$ be the predicate

$$(\forall y)(\forall z)[C(x, y, z) \wedge (y \neq 0) \Rightarrow |u - \rho(z)| < y^{-1}],$$

with the quantifiers "$\forall y$" and "$\forall z$" relativized by $N(x)$. Clearly, $(\forall x)(\forall v)(\forall u)[A(x, u) \wedge A(x, v) \Rightarrow (u = v)]$ and $(\forall u)(\exists! x)A(x, u)$ (the quantifiers "$\forall u$" and "$\exists! x$" being relativized by $N(x)$) hold in all subfields of $\mathcal{R}^C$. For $t_0 \in \mathcal{N}_0$ and $\alpha \in \mathcal{R}^C$, $A(t_0, \alpha)$ expresses "there is a 'total partial recursive' function with Gödel number t_0 which computes α". Of course, the predicate $A(t_0, x)$ defines α in $\mathcal{R}^C$.

Let $I = \{x \in \mathcal{N}_0 | (\exists z)A(x, z)$ holds in $\mathcal{R}_0\}$. We claim that I is an arithmetical set. Choose an admissible indexing ϕ which renders $\mathcal{R}_0$ computable. Clearly, $\phi | \mathcal{N}_0$ is an arithmetical function. Let $\overline{A}(x, z)$ be the predicate resulting from relativising $A(x, z)$ by the predicate $\overline{R}(x)$ (where $\overline{R}(x)$ is the arithmetical predicate which defines the set $\phi(\mathcal{R}_0)$) and replacing the relational symbols $S(x, y, z)$, $P(x, y, z)$, $Q(x, z)$, and $N(x)$ by the predicates $\overline{S}(x, y, z)$, $\overline{P}(x, y, z)$, $\overline{Q}(x, z)$, and $\overline{N}(x)$ (i.e., the arithmetical relations of the arithmetical representation of $\mathcal{R}_1$).

Use $\overline{I}$ for $\phi(I)$. $\overline{I}$ is arithmetical, since $x \in \overline{I} \equiv$ '$\overline{N}(x) \wedge (\exists z)[\overline{R}(z) \wedge \overline{A}(x, z)]$ holds in $\mathcal{N}_0$'. Use $\overline{I}(x)$ for "$x \in \overline{I}$".

Now, $x \in I \equiv$ '$(\exists z)A(x, z)$ holds in $\mathcal{R}_1 \equiv (\exists y)(\exists z)[\phi(x) = y \wedge \overline{A}(x, z)]$ holds in $\mathcal{N}_0$'. Use $I(x)$ for "$x \in I$". At this point we can observe that $x \in R_1 \equiv$ '$(\exists y)[I(y) \wedge A(y, x)]$ holds in $\mathcal{R}^C$.' This is to say $\mathcal{R}_1$ is arithmetically definable in $\mathcal{R}^C$, hence in $\mathcal{R}$, where our defining predicate is $(\exists y)[I(y) \wedge A(y, x)]$ which we choose to denote by $H(x)$. To complete our proof we simply continue as in the proof of Theorem 6. Our theorem is proved.

Theorem 8. *Let* $\mathcal{R}_1 \subseteq \mathcal{R}^C$, $\mathcal{R}^* = \{R^*; \mathcal{N}^*\}$ *a non-standard model of analysis. Every elementary extension* ${}^*\mathcal{R} = \{{}^*R; \mathcal{N}^*\}$ *of* $\mathcal{R}_1$ *is isomorphic to a subfield of* $\mathcal{R}^*$.

Proof. Consider the predicate $A(x, u)$ of the proof of Theorem 7. Since $\mathcal{R}_1 \vDash (\forall u)(\exists x)A(x, u)$, we have that ${}^*\mathcal{R} \vDash (\forall u)(\exists x)A(x, u)$; and so, for $\alpha \in {}^*\mathcal{R}$ there is a $t_0 \in \mathcal{N}^*$ such that ${}^*\mathcal{R} \vDash A(t_0, \alpha)$. Let $\psi_{t_0}: \mathcal{N}^* \to \mathcal{N}^*$ be defined by $\varphi_{t_0}(a) = b \equiv \mathcal{N}^* \vDash C(t_0, a, b)$. Further, let $A = \{\rho^*(\psi_{t_0}(a)) - a^{-1} | (a \neq 0) \wedge (a \in \mathcal{N}^*)\}$, where ρ^* is the natural extension of ρ to $\mathcal{N}^*$.

We claim that $\sup^{R^*} A$ exists. Let $b = \lceil \alpha \rceil + 1$, where $\lceil \alpha \rceil$ denotes greatest element of $\mathcal{N}^*$ in α. So $b \in \mathcal{N}^*$. Now if $T(u, v)$ denotes the predicate $(\forall x)(\forall y)[C(u, x, y) \wedge (\rho^*(y) - x^{-1}) = v]$ then two facts are immediate:

4.1) $r \in A \equiv Q^* \vDash T(t_0, r)$,
4.2) $Q^* \vDash (\forall v)[T(t_0, v) \Rightarrow (v \leq b)]$.

Moreover, since there is a predicate $Q(x)$ defined in $\Sigma_{\mathcal{R}_1}$ such that $r \in Q^* \equiv \mathcal{R}^* \vDash Q(r)$, we have

4.3) $r \in A \equiv \mathcal{R}^* \vDash T(t_0, r)$,
4.4) $\mathcal{R}^* \vDash (\forall v)[T(t_0, v) \Rightarrow (v \leq b)]$.

Now, if $U(u, v)$ is any predicate (with no individual constants) and X is the sentence

$$(\forall x)(\forall y)[(\forall u)[\mathsf{U}(x, u) \Rightarrow (u \leq y)] \Rightarrow (\exists!\, w)(\forall u)[[\mathsf{U}(x, u) \Rightarrow (u \leq w)] \wedge (\forall v)[\mathsf{U}(x, u) \Rightarrow (u \leq v)] \Rightarrow (w \leq v)]],$$

then $\mathcal{R} \vDash X$. Thus $\mathcal{R}^* \vDash X$. It follows that

$$\mathcal{R}^* \vDash (\forall u)[\mathsf{T}(t_0, u) \Rightarrow (u \leq b)] \Rightarrow (\exists!\, w)(\forall u)[[\mathsf{T}(t_0, u) \Rightarrow (u \leq w)] \wedge (\forall v)[\mathsf{T}(t_0, u) \Rightarrow (u \leq v)] \Rightarrow (w \leq v)]$$

and so

$$^*\mathcal{R} \vDash (\exists!\, w)(\forall u)[[\mathsf{T}(t_0, u) \Rightarrow (u \leq w)] \wedge (\forall v)[\mathsf{T}(t_0, u) \Rightarrow (u \leq v)] \Rightarrow (w \leq v)].$$

This is to say, $\sup^{R^*} \mathsf{A}$ exists.

One can define a function λ: $^*R \to R^*$ as follows: Given $\alpha \in {}^*R$, choose A as above then $\lambda(\alpha) = \sup^{R^*}\mathsf{A}$. Our claim is that λ is a monomorphism. First of all, it is known that for any ordered group, if $\sup C$ and $\sup D$ exist then $\sup(C + D) = \sup C + \sup D$. Thus, let $\alpha, \beta \in {}^*R$. Then determine sets A and B, corresponding to α and β, respectively. Thus

$$\lambda(\alpha) + \lambda(\beta) = \sup^{R^*}A + \sup^{R^*}B = \sup^{R^*}(A + B)$$

and $\sup^{R^*}(A + B)$ is easily seen to be equal to $\lambda(\alpha + \beta)$. Also, it is clear that if $\alpha \leq \beta$ then $\lambda(\alpha) \leq \lambda(\beta)$. In summary, the ordered commutative group of $^*\mathcal{R}$ is embedded in $\mathcal{R}^*$ via λ.

It suffices to show that $\lambda(\alpha\beta) = \lambda(\alpha)\lambda(\beta)$. This also follows along the lines of a very standard proof. For, let $\alpha \in {}^*R$ and $\alpha \neq 0$; denote by T_α the function defined by the equation $\mathsf{T}_\alpha(\beta) = \alpha\beta$ (i.e., T_α: $^*R \to {}^*R$). It is known that for any ordered group T_α there is an automorphism, $\mathsf{T}_{\alpha^{-1}} = (\mathsf{T}_\alpha)^{-1}$ and $\lambda \circ \mathsf{T}_\alpha = \mathsf{T}_{\lambda(\alpha)} \circ \lambda$.

Let $\alpha > 0$. Then $\lambda(\alpha\beta) = \lambda \circ (\mathsf{T}_\alpha \circ \mathsf{T}_\alpha^{-1})(\alpha\beta) = \lambda \circ (\mathsf{T}_\alpha \circ \mathsf{T}_{\alpha^{-1}})(\alpha\beta) = (\lambda \circ \mathsf{T}_\alpha)(\beta) = \mathsf{T}_{\lambda(\alpha)} \circ \lambda(\beta) = \mathsf{T}_{\lambda(\alpha)}(\lambda(\beta)) = \lambda(\alpha)\lambda(\beta)$. For $\alpha < 0$, $-\alpha > 0$; therefore, since $\lambda(-x) = -\lambda(x)$, we have $\lambda(\alpha\beta) = -\lambda(-\alpha)\lambda(\beta) = \lambda(\alpha)\lambda(\beta)$. Of course, if $\alpha = 0$, then $\lambda(\alpha\beta) = \lambda(0) = \lambda(0)\lambda(\beta)$. Our theorem is proved.

Corollary 9. *Let* $\mathcal{R}_1$ *be a computable ordered subfield of* $\mathcal{R}$, $\mathcal{R}^* = \{R^*; \mathcal{N}^*\}$ *be a non-standard model of analysis. Every elementary extension* $^*\mathcal{R} = \{^*R; \mathcal{N}^*\}$ *of* $\mathcal{R}_1$ *is isomorphic to a subfield of* $\mathcal{R}^*$.

Proof. Every computable ordered subfield of $\mathcal{R}$ is properly contained in $\mathcal{R}^C$. See [1].

Remark. It is worth noting that the result of Theorem 7 becomes a consequence of Theorem 8, when one recalls from [2] that given any AD-structure $\mathcal{S} = \{S, \mathcal{N}\}$ and any non-standard model of arithmetic $\mathcal{N}^*$ there is an elementary extension of $\mathcal{S}$, say $\mathcal{S}^* = \{S^*; \mathcal{N}^*\}$ (i.e., an elementary extension of $\mathcal{S}$ determined by $\mathcal{N}^*$). Then use Theorem 8 to embed $^*\mathcal{R}$ in $\mathcal{R}^*$.

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The University of Iowa
Iowa City, Iowa