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A NOTE ON Π_1^1 ORDINALS

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In reference [3] Tanaka proves (among other things) that the Π_1^1 ordinals are precisely the ordinals recursive in Kleene's set O.[1] The purpose of this note is to show how this result may be neatly obtained as a corollary of reference [2]. Here is some background on the matter.

An ordinal α is called *recursive* (Σ_1^1, Π_1^1, recursive in O, etc.) if there is a recursive (Σ_1^1, Π_1^1, recursive in O, etc.) well-ordering of natural numbers with order type α. O is the set of notations of Kleene's system S_3, and ω_1 (ω_1^{O}) is the least ordinal that is not recursive (recursive in O). Some well-known facts of ordinal notation theory are the following, where each set is an initial segment of ordinals.

(1) $\{\alpha : \alpha \textit{ is recursive}\} = \{\alpha : \alpha \textit{ is } \Sigma_1^1\} \subset \{\alpha : \alpha \textit{ is } \Pi_1^1\} \subseteq \{\alpha : \alpha \textit{ is recursive in } \mathrm{O}\}$.

Tanaka's result concerns the final inclusion in (1):

PROPOSITION. $\{\alpha : \alpha \textit{ is } \Pi_1^1\} = \{\alpha : \alpha \textit{ is recursive in } \mathrm{O}\}$

Proof, derived from [2].[2] We show that every ordinal less than ω_1^{O} is Π_1^1. In the notation of [2], $\mathsf{W}[A]$ is the set of all natural numbers e such that the partial recursive function $\{e\}$ is defined on $A \times A$, and $\{(x, y) : \{e\}(x, y) = 0\}$ well-orders A. If A is infinite, then the order types of such well-orderings comprise a segment of ordinals beginning with ω. The least upper bound of the segment is denoted by "$|\mathsf{W}[A]|$". Remark 4.8 and theorem 7.3 of [2] imply that

(2) $|\mathsf{W}[\mathrm{O}]| = \omega_1^{\mathrm{O}}$,

which is exactly the needed fact:

1. This fact is also proved in § VI.1 of [1].
2. This proof, more direct than the one appearing in [1], was suggested to the author by Professor Richter.

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If α is an infinite ordinal less than ω_1^{O}, then, by (2), *there is an* $e \in \mathrm{W}[\mathrm{O}]$ *such that α is the order type of the Π_1^1 well-ordering* $\{(x, y) : x \in \mathrm{O} \;\&\; y \in \mathrm{O} \;\&\; \{e\}(x, y) = 0\}$. Q.E.D.

To generalize the proposition, we set $\mathrm{O}(0) =_{def} \phi$ and $\mathrm{O}(n+1) =_{def} \mathrm{O}^{\mathrm{O}(n)}$. Then one may prove by induction, beginning with (2), that

$$\{\alpha : \alpha \textit{ is } \Pi_1^{1,\mathrm{O}(n)}\} = \{\alpha : \alpha \textit{ is recursive in } \mathrm{O}(n+1)\}.$$

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