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HENKIN STYLE COMPLETENESS PROOFS IN THEORIES LACKING NEGATION

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As they are customarily formulated, Henkin style completeness proofs are not applicable to logical theories lacking negation. The purpose of this note is to show that if such a theory contains disjunction, either as a primitive logical constant or as a defined one, then a slight modification of the ordinary constructions can be used to construct a completeness proof. This procedure yields completeness proofs for a large group of truth-functionally incomplete propositional calculi. Many of these completeness results are already known, but this procedure yields a much simpler proof than the customary ones, and establishes all of these results simultaneously rather than piecemeal. The procedure can also be used in more complex cases, for example, first-order theories lacking negation.

1. We assume that the rules of whatever logical theory is being investigated are such that the relation "$\rightarrow$" of deducibility satisfies the following conditions:

(P) If $P \epsilon \Gamma$ then $\Gamma \rightarrow P$.
(trans) If $\Gamma \subseteq \Lambda$ and $\Gamma \rightarrow P$ then $\Lambda \rightarrow P$.
(AE) If $\Gamma \rightarrow P \vee Q$, $\Lambda \cup \{P\} \rightarrow R$, and $\Phi \cup \{Q\} \rightarrow R$, then $\Gamma \cup \Lambda \cup \Phi \rightarrow R$.
(AI) If $\Gamma \rightarrow P$ then $\Gamma \rightarrow P \vee Q$ and $\Gamma \rightarrow Q \vee P$.

Let '$P_1 \vee \ldots \vee P_n$' be an abbreviation for '$P_1 \vee (P_2 \vee (P_3 \vee \ldots \vee P_n))$'. From (P), (trans), (AE), and (AI) we easily obtain:

(1) If $\Gamma \rightarrow P \vee R$ and $\Gamma \cup \{P\} \rightarrow Q$, then $\Gamma \rightarrow Q \vee R$.
(2) If $\Gamma \rightarrow (P_1 \vee \ldots \vee P_n)$ then $\Gamma \rightarrow P_i \vee P_1 \vee \ldots \vee P_{i-1} \vee P_{i+1} \vee \ldots \vee P_n$.
(3) If $\Gamma \rightarrow (P_1 \vee \ldots \vee P_n) \vee (Q_1 \vee \ldots \vee Q_m)$ then $\Gamma \rightarrow P_1 \vee \ldots \vee P_n \vee Q_1 \vee \ldots \vee Q_m$.

2. Our objective is to prove that for all Γ, P, if $\Gamma \Rightarrow P$ (Γ truth-functionally implies P) then $\Gamma \rightarrow P$. The strategy is to prove the contrapositive. We suppose that $\Gamma \not\rightarrow P$. Consider an enumeration P_n of the formulas of the language. Define '$\Gamma \rightarrow \Sigma\Lambda$' to mean 'there are $R_1, \ldots, R_k \epsilon \Lambda$ such that $\Gamma \rightarrow R_1 \vee \ldots \vee R_k$'. Then define inductively:

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$$A_0 = \Gamma \qquad\qquad B_0 = \{P\}.$$

$$A_{n+1} = \begin{cases} A_n \cup \{P_n\} \text{ if } A_n \cup \{P_n\} \not\to \Sigma B_n \\ A_n \text{ otherwise} \end{cases} \qquad B_{n+1} = \begin{cases} B_n \text{ if } P_n \epsilon A_{n+1} \\ B_n \cup \{P_n\} \text{ otherwise} \end{cases}$$

$$A = \bigcup_{n\epsilon\omega} A_n \qquad\qquad B = \bigcup_{n\epsilon\omega} B_n$$

This replaces the ordinary Henkin construction which requires negation. The idea is that A contains true formulas and B contains false formulas. The objective is now to show that there is a truth value assignment V resulting in just these sets of true and false formulas, and hence satisfying Γ but making P false. Such an assignment is obtained by defining, for atomic formulas R, $V(R) = T$ if $R\epsilon A$ and $V(R) = F$ otherwise. It must be shown that for arbitrary Q, Q is true under V iff $Q\epsilon A$.

By the construction, $P_n\epsilon A$ iff $P_n\epsilon A_{n+1}$, $P_n\epsilon B$ iff $P_n\epsilon B_{n+1}$, and $P_n\epsilon A_{n+1}$ iff $P_n\not\epsilon B_{n+1}$, so:

Lemma 1: *$Q\epsilon A$ iff $Q\not\epsilon B$.*

Lemma 2: $A \not\to \Sigma B$.

We prove by induction that $A_n \not\to \Sigma B_n$: By supposition, $A_0 \not\to P$. Suppose $A_n \not\to \Sigma B_n$, but $A_{n+1} \to \Sigma B_{n+1}$. Then there are $R_1, \ldots, R_k \epsilon B_{n+1}$ such that $A_{n+1} \to R_1 \vee \ldots \vee R_k$. Suppose $P_n\epsilon A_{n+1}$. Then $B_{n+1} = B_n$, and $A_{n+1} = A_n \cup \{P_n\}$, so $A_n \cup \{P_n\} \to \Sigma B_n$. But then, by the construction of A_{n+1} and contrary to our supposition, $P_n\not\epsilon A_{n+1}$. So $P_n\not\epsilon A_{n+1}$. Thus $A_{n+1} = A_n$, and $A_n \to R_1 \vee \ldots \vee R_k$. As $A_n \not\to \Sigma B_n$, one of the R_i is not in B_n, and so must be P_n. Thus by (2), $A_n \to P_n \vee R_1 \vee \ldots \vee R_{i-1} \vee R_{i+1} \vee \ldots \vee R_k$. By the definition of A_{n+1}, as $P_n\not\epsilon A_{n+1}$, we must have $A_n \cup \{P_n\} \to S_1 \vee \ldots \vee S_m$ for some $S_1, \ldots, S_m \epsilon B_n$. Thus by (1), $A_n \to (S_1 \vee \ldots \vee S_m) \vee (R_1 \vee \ldots \vee R_{i-1} \vee R_{i+1} \vee \ldots \vee R_k)$, and hence by (3), $A_n \to S_1 \vee \ldots \vee S_m \vee R_1 \vee \ldots \vee R_{i-1} \vee R_{i+1} \vee \ldots \vee R_k$, i.e., $A_n \to \Sigma B_n$. But this contradicts the induction hypothesis, so $A_{n+1} \not\to \Sigma B_{n+1}$.

Therefore, for each n, $A_n \not\to \Sigma B_n$. But if $A \to \Sigma B$, then for some n, m, $A_n \to \Sigma B_m$. By (trans), letting k be the maximum of n, m, we have $A_k \to \Sigma B_k$. Consequently, $A \not\to \Sigma B$.

Lemma 3: *If $A \to Q$ then $Q\epsilon A$.*

Proof: Suppose $A \to Q$ but $Q\not\epsilon A$. Then by lemma 1, $Q\epsilon B$. Then as $A \to Q$, $A \to \Sigma B$, which contradicts lemma 2.

To complete the proof that for each R, $R\epsilon A$ iff R is true under V, we need lemmas like:

(i) $(P \,\&\, Q)\epsilon A$ iff $P\epsilon A$ and $Q\epsilon A$.
(ii) $(P \vee Q)\epsilon A$ iff $P\epsilon A$ or $Q\epsilon A$.
(iii) $(P \supset Q)\epsilon A$ iff $P\not\epsilon B$ or $Q\epsilon A$.
(iv) $(P \equiv Q)\epsilon A$ iff, $P\epsilon A$ iff $Q\epsilon A$.

Which lemmas we need depends of course upon which connectives we have in our language. Given the appropriate lemmas, the completeness theorem follows immediately. (ii) follows easily from (P), (AE), and (AI). The others can be obtained by employing the usual introduction and elimination rules:

(KE) $\dfrac{\Gamma \to P \mathbin{\&} Q}{\Gamma \to P \text{ and } \Gamma \to Q}$ (KI) $\dfrac{\Gamma \to P \text{ and } \Lambda \to Q}{\Gamma \cup \Lambda \to P \mathbin{\&} Q}$

(CE) $\dfrac{\Gamma \to P \supset Q \text{ and } \Lambda \to P}{\Gamma \cup \Lambda \to Q}$ (CI) $\dfrac{\Gamma \cup \{P\} \to Q}{\Gamma \to P \supset Q}$

(EE) $\dfrac{\Gamma \to P \equiv Q \quad \Lambda \to P}{\Gamma \cup \Lambda \to Q}$ $\dfrac{\Gamma \to P \equiv Q \quad \Lambda \to Q}{\Gamma \cup \Lambda \to P}$ (EI) $\dfrac{\Gamma \cup \{P\} \to Q \quad \Lambda \cup \{Q\} \to P}{\Gamma \cup \Lambda \to P \equiv Q}$

3. If '$\vee$' is one of the logical constants of a truth functional propositional calculus, then we can obtain a complete set of rules for that theory by adopting (P), (AE), (AI), and the introduction and elimination rules corresponding to whatever other logical constants are contained in the calculus. In each case it is easily verified that (trans) holds. If '$\vee$' is not one of the logical constants, but '$\supset$' is, then we can define '$P \vee Q$' to be '$(P \supset Q) \supset Q$' and the above results still hold. It is of interest however that we must still have (AE) as a rule (suitably transcribed in terms of '$\supset$'). (AI) can readily be derived from (CE) and (CI), but (AE) cannot. This can be seen by considering a three valued semantics:

		Q		
$P \supset Q$		0	1	2
	0	2	2	2
P	1	0	2	2
	2	0	1	2

$\Gamma \Rightarrow P$ iff for all assignments V, if $V(P) \neq 2$ then there is a $Q \epsilon \Gamma$ such that $V(Q) \leq V(P)$.

(P), (CE), and (CI) are sound in this semantics, but (AE) fails in the case where $\Gamma = \{(P \supset Q) \supset Q\}$, $\Lambda = \{P \supset R\}$, $\Phi = \{Q \supset R\}$, and $V(P) = 1$, $V(Q) = 0$, $V(R) = 1$. However, it follows from the above that (P), (AE), (CE), and (CI) gives us a complete purely implicational propositional calculus.

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