Notre Dame Journal of Formal Logic
Volume XII, Number 4, October 1971
NDJFAM

A PROOF OF A THEOREM OF ŁUKASIEWICZ

IVO THOMAS

We present a proof of Łukasiewicz's theorem in [1] that Syllogism, Peirce and any expression of the form $CpC\alpha\beta$ are, with detachment and substitution, sufficient for two-valued implication. Łukasiewicz proceeded by deriving $CpCqp$ with masterly economy of detachments, to get the Bernays axioms. The present method obtains instead $CpCCqqCqq$ to get the Wajsberg axioms from [2]. Though greatly more prodigal of detachments, it seems pleasing in virtue of a certain simplicity in the use of auxiliary theses derived only from 1 and 2, and by its delay of the use of 3 till the last moment. It may perhaps be found easier to remember than Łukasiewicz's and to be useful for purposes of instruction. The axioms are:

1. $CCpqCCqrCpr$
2. $CCCpqpp$
3. $CpC\alpha\beta$

From 1 and 2 we derive:

$D1.1 = 4.$	$CCCCqrCprsCCpqs$
$D4.4 = 5.$	$CCpCqrCCsqCpCsr$
$D4D5.1 = 6.$	$CCpqCCspCCqrCsr$
$D4.2 = 7.$	$CCpCpqCpq$
$D1.2 = 8.$	$CCprCCCpqpr$
$D6.2 = 9.$	$CCsCCpqpCCprCsr$
$D1D4.8 > 10.$	$CCCRRsCCCRQRs$
$D1DD1.6.8 > 11.$	$CCCCRQRsCQs$

10 and 11 are not the most general results of the detachments but substitutions in those. Q abbreviates Cqq, R abbreviates CQQ. 4 and 8 will not be used again, 6 only once.

In the following list, each member implies its successor in accordance with the thesis mentioned, the first therefore implies the last by successive applications of 1.

Received October 25, 1970

$C\alpha\beta$

$CCCRQ\alpha CC\beta QCCRQQ$ (6)

$\underbrace{\qquad\qquad\qquad\qquad}_{B}$

$CCBCRQCCCRQ\alpha CRQ$ (1)

$\underbrace{\qquad}_{B'}$

$CCCRQQCB'Q$ (9)

$CCRB'CCCRQQCRQ$ (5)

$CCCRQRCCRB'R$ (9)

$CCRRCCCRQRR$ (9)

$CCCRQRCCCRQRR$ (10)

$CCCRQRR$ (7)

CQR (11)

R (7)

We have thus proved $CC\alpha\beta CCqqCqq$, whence by 1 and 3 we have the Wajsberg axiom $CpCCqqCqq$, and the proof is complete.

BIBLIOGRAPHY

[1] Łukasiewicz, J. W., "W Sprawie Aksjomatyki Implikacyjnego Rachunku Zdań," (Concerning an axiom system of the implicational propositional calculus). *VI Zjazd Matematyków Polskich*, Warszawa, September 20-23, 1948. Supplement to *Annales de la Société Polonaise de Mathématique*, Kraków, vol. 22 (1950), pp. 87-92.

[2] Wajsberg, M., "Contributions to Metalogic—II," *Th.16*, in *Polish Logic* 1920-1939, Storrs McCall, Ed., Oxford (1967).

University of Notre Dame
Notre Dame, Indiana