Notre Dame Journal of Formal Logic
Volume XII, Number 4, October 1971
NDJFAM

SOME RESULTS ON GENERALIZED TRUTH-TABLES

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A generalized truth-table (also called a model) is an algebraic structure $\mathfrak{M} = (A, D, \Omega)$ such that A is a non-empty set of elements, D is a subset of A and Ω is a non-empty finite set of n-ary operations (with n positive) defined on A. With each model (A, D, Ω) is associated the algebra (A, Ω), and (A, Ω) is called a *full algebra* if and only if (iff) for each $\omega \in \Omega$ the range of ω is A itself. A model (A, D, Ω) is called *full* iff its associated algebra is full. The usual two-valued truth-table for the classical propositional calculus is full. An example of a truth-table which is not full is:

C	0	1	2
0	1	1	1
*1	0	1	0
*2	1	1	1

The starred elements are designated. A model $\mathfrak{M}'$ is called a *super model* of a model $\mathfrak{M}$ iff every well-formed formula (wff) valid in $\mathfrak{M}$ is also valid in $\mathfrak{M}'$. Two models are called *equivalent* iff each is a super model of the other.

A propositional calculus P can sometimes be extended to another propositional calculus P' in such a manner that both P and P' have the same class of full models. An example is the pure implicational fragment $S4_I$ of Lewis's S4 and the intuitionist implicational calculus H_I. Also the intuitionist propositional calculus H, the classical propositional calculus K and all intermediate calculi have the same class of full models in which modus ponens is valid. A study of such extensions is made in [4] and [5], and in such a study it is sometimes important to know whether or not an arbitrary finite model has an equivalent full model. This problem is fully solved here for finite models with one operation and is partly solved in the general case. We include applications of our results to some well-known propositional calculi.

For the purpose of this paper we take a fixed but arbitrary language L_0 of order zero with the propositional variables $p_1, p_2, p_3, \ldots$, a non-empty

Received October 2, 1969

set Ω^* of n-ary propositional connectives (with n positive), and the set Γ of all wffs of L_0 defined in the usual way using Łukasiewicz notation. The lower case Greek letters α, β, γ, and δ, with or without subscripts will be used to denote arbitrary wffs. If $v_1, v_2, \ldots, v_k$ are precisely the propositional variables occurring in a wff α, then we sometimes write α as $\alpha(v_1, v_2, \ldots, v_k)$. If a wff β arises from $\alpha(v_1, v_2, \ldots, v_k)$ by the usual process of substitution of γ_i for v_i, then we write $\beta = \alpha(v_1/\gamma_1, v_2/\gamma_2, \ldots, v_k/\gamma_k)$. α is called *variable-like* iff no propositional variable occurs more than once in α. In the classical propositional calculus *CNpCqNr* is variable-like, but *CCpqCNqNp* is not variable-like. If $\alpha = \beta(v_1/v_{k+1}, v_2/v_{k+2}, \ldots, v_k/v_{2k})$ and $\beta = \alpha(v_{k+1}/v_1, v_{k+2}/v_2, \ldots, v_{2k}/v_k)$, then α and β are called *variants* of each other.

We confine our attention to models (A, D, Ω) such that (Γ, Ω^*) and (A, Ω) are similar as algebras, see [6]. For each model (A, D, Ω) and each α, $\alpha(v_1, v_2, \ldots, v_k)$ will be regarded as a function from A^k to A, and for each subset B of A,

$$\alpha(B) = \{x \in A \mid x = \alpha(y_1, y_2, \ldots, y_k) \textit{ for some } y_1, y_2, \ldots, y_k \in B\}.$$

$\alpha(A)$ is also called the *range of* α. A model (A, D, Ω) is *reducible* to a model (A_1, D_1, Ω) iff (A_1, Ω) is a full sub-algebra of (A, Ω), $D_1 = A_1 \cap D$ and there is a variable-like wff α such that $\alpha(A) \subseteq A_1$. A model is called *reducible* iff it is reducible to at least one model. The example of a non-full truth-table given earlier reduces to the classical implicational truth-table by the variable-like wff *Cpq*. We now prove a preliminary lemma needed in the applications of our results.

Lemma. *If* $\mathfrak{M}$ *and* $\mathfrak{N}$ *are models which reduce respectively to* $\mathfrak{M}'$ *and* $\mathfrak{N}'$, *then the direct product model* $\mathfrak{M} \times \mathfrak{N}$ *reduces to* $\mathfrak{M}' \times \mathfrak{N}'$.

Proof: The direct product of full models is full. Let α and β be variable-like wffs such that $\alpha(A) \subseteq A'$, $\beta(B) \subseteq B'$, where $\mathfrak{M} = (A, D, \Omega)$, $\mathfrak{M}' = (A', D', \Omega)$, $\mathfrak{N} = (B, D_1, \Omega_1)$ and $\mathfrak{N}' = (B', D_1', \Omega_1)$. Let $\gamma = \alpha(v_1/\beta^1, v_2/\beta^2, \ldots, v_k/\beta^k)$, where β^i, β^j for $i \neq j$ are variants of β and do not share a variable. Then $\gamma(A \times B) = \alpha(\beta(A \times B)) \subseteq A' \times B'$.

Remark: The preceding lemma is true for the direct product of an arbitrary family of models provided the same variable-like wff reduces each member of the family.

Theorem 1. *There is an effective method for deciding whether or not a finite model is reducible.*

Proof: First we show that there is an effective method for deciding whether or not any given subset of a finite model is the range of some variable-like wff α and if so to construct one such α.

Let (A, D, Ω) be a finite model. Consider the finite algebra (A, Ω). In order to carry out the proof we define inductively the sets V^n, such that for each n, V^n will contain the ranges of all variable-like wffs with at most n connectives.

Let $V^0 = \{A\}$, and for each k-ary $\omega \in \Omega$,
Let $V^n_\omega = \{X \subseteq A \mid X = \omega(Y_1 \times Y_2 \times \ldots \times Y_k)$ *for some* $Y_1, Y_2, \ldots, Y_k \in V^{n-1}\}$, and
Let $V^n = V^{n-1} \cup \bigcup_{\omega \in \Omega} V^n_\omega$.

By a simple inductive argument on the number of connectives in a variable-like wff one can show that the range of a variable-like wff with n connectives is in V^n. Clearly, $V^n \subseteq V^{n+1}$ for $n \geq 0$. Since A is finite, the sequence $V^0, V^1, \ldots$ can contain only finitely many distinct terms. Let m be the smallest non-negative integer such that $V^m = V^{m+r}$ for some $r > 0$. Since we have a nested sequence, $V^m = V^{m+1}$. Hence by the definition of V^n, $V^m = V^{m+i}$ for all $i > 0$. In fact, $m < 2^j$, where A has j elements.

Since the full sub-algebras, if any, of a finite algebra can be effectively listed, the preceding proof gives an effective method for checking whether or not the range of some variable-like wff is contained in one of the full sub-algebras. For each element of V^i one can construct by using the definition of V^i a variable-like wff whose range coincides with this element.

Remark: Note that if (A, Ω) is full, then $V^0 = V^1$, and hence the range in (A, Ω) of every variable-like wff is A.

Theorem 2. *Every finite model with exactly one operation is reducible.*

Proof: Let $(A, D, \{\omega\})$ be a finite model, where ω is a k-ary operation. Define $A_0 = A$ and for each $n > 0$ let $A_n = \omega(A_{n-1})$. It follows by induction on n that $A_{n+1} \subseteq A_n$ for each $n \geq 0$. Since the A_n form a decreasing sequence of non-empty subsets of a finite set, we have, for some r, $A_r = A_{r+1}$, i.e., $A_r = \omega(A_r)$. Thus A_r with ω restricted to it is a full sub-algebra of $(A, \{\omega\})$.

Let $\Omega^* = \{\omega^*\}$, where ω^* is a k-ary propositional connective. Let $\alpha_0 = p_1$ and for $n > 0$, let $\alpha_n = \omega^*(\alpha^1_{n-1}, \alpha^2_{n-1}, \ldots, \alpha^k_{n-1})$, where each α^i_{n-1} is a variant of α^j_{n-1} and $\alpha^i_{n-1}, \alpha^j_{n-1}$ do not share a propositional variable for $i \neq j$. Thus, $\alpha_0, \alpha_1, \alpha_2, \ldots$ is a sequence of variable-like wffs. Moreover, it is easily verified by induction on n that $\alpha_n(A) = A_n$ for each $n \geq 0$. Thus, $(A, D, \{\omega\})$ is reducible to $(A_r, A_r \cap D, \{\omega\})$, and α_r is a required variable-like wff.

If $\alpha = \beta(v_1/\delta_1, v_2/\delta_2, \ldots, v_k/\delta_k)$, each δ_i is a variable-like wff and no two distinct δ_i share a variable, then β is called a *restricted generalization* of α. Every wff is a restricted generalization of any variant of it, and $CpCqp$ and $CCpCqrCqCpr$ are restricted generalizations respectively of $CCpqCrCpq$ and $CCpCCqrsCCqrCps$.

We now prove a theorem concerning restricted generalizations which is of some independent interest and which is needed in the proof of our main theorem.

Theorem 3. *Let* $\mathfrak{M}$ *be a full model and let* β *be a restricted generalization of* α, *then* α *is valid in* $\mathfrak{M}$ *iff* β *is valid in* $\mathfrak{M}$.

Proof: In a full model (A, D, Ω) the range of every variable-like wff is A itself. (See remark in the proof of Theorem 1.)

Let $\alpha = \beta(v_1/\delta_1, v_2/\delta_2, \ldots, v_k/\delta_k)$, where each δ_i is variable-like and no two distinct δ_i share a variable. Since α is a substitution instance of β, $\alpha(A) \subseteq \beta(A)$. To prove the converse inclusion we note that each δ_i is variable-like and hence has A for its range. Since no two distinct δ_i share a variable they can be made to take simultaneously any arbitrary values. Thus, for any value that β takes we can find an assignment under which α takes the same value.

Theorem 4. *If a model $\mathfrak{M}$ is reducible to a model $\mathfrak{M}'$, then $\mathfrak{M}'$ is unique and every full super model of $\mathfrak{M}$ is a full super model of $\mathfrak{M}'$.*

Proof: Let $\mathfrak{M} = (A, D, \Omega)$ be reducible to $\mathfrak{M}' = (B, B \cap D, \Omega)$. Then (B, Ω) is a full sub-algebra of (A, Ω) and there is a variable-like wff β such that $\beta(A) \subseteq B$.

We first show the uniqueness of (B, Ω). Let (C, Ω) be any full sub-algebra of (A, Ω) and let α be any variable-like wff. Since α can be interpreted as a function in (C, Ω) which is full, $\alpha(C) = C$ by the remark in the proof of Theorem 1. Since $C \subseteq A$, $\alpha(C) = C \subseteq \alpha(A)$. We also have in particular $\beta(A) = B$. Let γ and (C, Ω) be any variable-like wff and full sub-algebra of (A, Ω) such that $\gamma(A) \subseteq C$. Since (C, Ω) and (B, Ω), are full sub-algebras of (A, Ω) and γ and β are variable-like wffs $C \subseteq \beta(A)$ and $B \subseteq \gamma(A)$. Therefore, $B = C$.

Clearly, $\mathfrak{M}' = (B, B \cap D, \Omega)$ is a full super model of $\mathfrak{M}$. We now proceed to show the minimality of $\mathfrak{M}'$. We first show that every wff valid in $\mathfrak{M}'$ is a restricted generalization of some wff valid in $\mathfrak{M}$. Let α be valid in $\mathfrak{M}'$. Consider the substitution T_β such that $T_\beta(\alpha) = \alpha^* = \alpha(v_1/\beta^1, v_2/\beta^2, \ldots, v_k/\beta^k)$, where each β^i is a variant of β^j and for $i \neq j, \beta^i$ and β^j do not share a variable. Then, α is a restricted generalization of α^* and α^* is valid in $\mathfrak{M}$, for otherwise there is an assignment for the variables of α^* such that α^* takes a value $d \epsilon A - D$. Letting b_i be the value of β^i for this assignment and using the fact that $\beta(A) = \beta^i(A) = B$, we have $\alpha(b_1, b_2, \ldots, b_k) = d \notin B \cap D$. But this contradicts the validity of α in $\mathfrak{M}'$. Now, let $\mathfrak{M}_2$ be any full super model of $\mathfrak{M}$. Since $\mathfrak{M}_2$ is full any restricted generalization of any valid wff of $\mathfrak{M}_2$ is also valid in $\mathfrak{M}_2$ by Theorem 3. Since $\mathfrak{M}_2$ is a super model of $\mathfrak{M}$, every restricted generalization of any wff valid in $\mathfrak{M}$ must also be valid in $\mathfrak{M}_2$. Hence $\mathfrak{M}_2$ is a full super model of $\mathfrak{M}'$.

Remark: If a model $\mathfrak{M}$ is finite and reducible, then the full super model $\mathfrak{M}'$ of Theorem 4 can be effectively determined by using the methods of Theorem 1. The test for equivalence of finite models due to Kalicki [3] can now be used to determine whether or not $\mathfrak{M}$ and $\mathfrak{M}'$ are equivalent. If $\mathfrak{M}$ and $\mathfrak{M}'$ are not equivalent, then in view of the minimality of $\mathfrak{M}'$ there is no full model equivalent to $\mathfrak{M}$.

The proof of the minimality of $\mathfrak{M}'$ in Theorem 4 implies the following

Corollary: *If $\mathfrak{M}$ is reducible to $\mathfrak{M}'$ then there is a substitution T_β such that α is valid in $\mathfrak{M}'$ iff $T_\beta(\alpha)$ is valid in $\mathfrak{M}$, where T_β is the substitution described in the proof of Theorem 4.*

Applications: The denumerable model $\mathfrak{M}$ of Dummett [2] characterizing LC is not full and is reducible to the usual two-valued model for classical propositional calculus K by the variable-like wff Np. Since LC is a proper subcalculus of K it follows by Theorem 4 that $\mathfrak{M}$ has no equivalent full model. Also by the Corollary, $\alpha(v_1, v_2, \ldots, v_n)$ *is a classical tautology iff* $\alpha(v_1/Nv_1, v_2/Nv_2, \ldots, v_n/Nv_n)$ *is a theorem of* LC.

Since $CNNNpNp$ is a theorem of the intuitionist propositional calculus H and its restricted generalization $CNNpp$ is not provable in H, it follows by Theorem 3 that H does not have a characteristic full model. The addition of $CNNpp$ to H gives K. Hence no intermediate propositional calculus (including LC) other than K has a full characteristic model.

Since p, App, Kpp, $CCppp$ are all equivalent in H, the usual Lindenbaum models of equivalence classes of wffs for the various negation free fragments of H and their extensions are full. By Theorem 3, the set of theorems of each of these calculi is closed under restricted generalization.

Here we establish another connection between theorems of H and K of the type described in [6, pp. 390-394]. If $\alpha(v_1, v_2, \ldots, v_n)$ *is any disjunction-free wff, then* $\alpha(v_1, v_2, \ldots, v_n)$ *is a classical tautology iff* $\alpha(v_1/Nv_1, v_2/Nv_2, \ldots, v_n/Nv_n)$ *is an intuitionist tautology.* This statement implies Theorem 5.9 of [6, p. 393]. Let $\mathfrak{J}_i'$ denote the model obtained from the Jaskowski model $\mathfrak{J}_i$ described in [7] by ignoring disjunction. Each such $\mathfrak{J}_i$ contains a full sub-model $\mathfrak{K}_i'$ consisting of precisely those elements of $\mathfrak{J}_i'$ all whose coordinates are either 0 or 1. This is verified by induction. $\mathfrak{K}_i'$ characterizes the set of all disjunction-free classical tautologies. Also each $\mathfrak{J}_i'$ is reducible to $\mathfrak{K}_i'$ by the variable-like wff Np. Hence by a remark following the Lemma $\Pi_{i=1}^{\infty} \mathfrak{J}_i'$ is reducible to $\Pi_{i=1}^{\infty} \mathfrak{K}_i'$. Now our statement follows by the corollary.

Let $\mathfrak{M}$ be an infinite characteristic matrix of Lewis's S5 satisfying the conditions of Theorem 6 of Scrogg [8]. Corresponding to strict implication we define in $\mathfrak{M}$, $x \dashv y = -*(x \times -y)$. $\mathfrak{M}$ is reducible to the two-valued truth-table $\mathfrak{M}'$ by the variable-like wff $p \dashv q$. In $\mathfrak{M}'$, $*$ acts as the identity operation. We note that the operation $\dashv$ restricted to $\mathfrak{M}'$ corresponds to material implication. Hence by the corollary, α *is a classical tautology iff* $S(\alpha)$ *is a theorem of* S5, where $S(\alpha)$ results from α by replacing each material implication sign by $\dashv$ leaving the other connectives unchanged and by replacing each variable v of α by a variant of $p \dashv q$ such that variants of $p \dashv q$ replacing distinct variables of α do not share a variable.

Let R denote the set of real numbers, let $\mathrm{P}(R)$ and $\mathrm{O}(R)$ denote respectively the power set and the class of all open sets of reals (in the usual topology). By Theorem 9.1 (vii) of [6, p. 478] the model $\mathfrak{M} = (\mathrm{P}(R), \{R\}, \{\cap, \cup, \dashv\})$ characterizes the corresponding fragment of S4, where $A \dashv B$ is the interior of $(R - A) \cup B$. $\mathfrak{M}$ is reducible to the model $\mathfrak{M}' = (\mathrm{O}(R), \{R\}, \{\cap, \cup, \dashv\})$ by the variable-like wff $p \dashv q$. By Theorem 3.2 (vii) of [6, p. 386] $\mathfrak{M}'$ characterizes the positive fragment of H. By statement (3) of [6, p. 59] the operation $\dashv$ restricted to $\mathfrak{M}'$ corresponds to intuitionist implication. *Thus, by our corollary, for any negation-free wff* α, α *is an intuitionist theorem iff* $S(\alpha)$ *is a theorem of* S4, where $S(\alpha)$ is as described in the preceding paragraph.

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