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THE STRONG DECIDABILITY OF CUT LOGICS
II: GENERALIZATIONS

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1. *Introduction.* In the first part of this paper [1] we discussed the result of restricting the number of applications of the rule modus ponens in a certain class of partial propositional calculi. Here we shall generalize the results of that paper to partial propositional calculi in general, and to the cut versions of modal logics.

We first recall some definitions. For details, the reader is referred to [1]. By a *partial propositional calculus*, we shall mean a triple $\langle M,R,N\rangle$ where M is a finite set of well-formed formulae which are theorems of the classical propositional calculus, R is either the rule modus ponens (MP) or the rule MP together with the rule simultaneous substitution (SS), or the rule MP together with the rule substitution (S), and N is a non-negative integer or infinity. If N is infinity, the calculus is to be thought of as the usual calculus with the rules R and the axioms M. The calculi with N finite are the same calculi with the restriction that the rule MP may be applied N times or fewer only. The calculi with N finite will be called *cut-calculi*. (This definition will be generalized for the case of modal logics.) Any calculus of either of the above types is called *decidable* if there exists an effective method, given any well-formed formula of the calculus, for deciding whether that formula is a theorem of the given calculus. Such a calculus will be called *strongly decidable* if one can effectively find a finite set of well-formed formulae such that the set of theorems of the calculus coincides with the set of SS instances of the given set of formulae.

2. *Partial propositional calculi.* In the first part of this paper, we restricted ourselves to calculi where implication ($\supset$) and a constant false (f) were the only connectives present. Here we place no restriction on what (finite number of) connectives and constants may be present but we do assume that $\supset$ is present, since we wish to have MP as one of our rules. (If this is not the case, the results still hold, but are not of as great interest.) We say that a singulary connective is of *degree* one, a binary connective is of degree two, etc., thus associating a finite integer with each connective. Since we are placing no restrictions on the number of times that the rule S

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may be used, for the sake of convenience in writing out proofs, we shall assume that the rules present are MP and SS (But see section 3.) The principal theorems below read exactly as they did in [1]. The difference is that the class of calculi to which these theorems apply is now considerably larger.

Theorem A. *For all M and for all finite N, ⟨M,R,N⟩ is strongly decidable; i.e., all cut propositional calculi are strongly decidable.*

Given a well-formed formula *W*, let *W*! be the collection of all SS instances of *W*. We call *W*! the *closure under* SS *of W*.

Theorem B. *For any two well-formed formulae W and X, the intersection of W! and X! is representable as a finite union, A!∪B!∪... ∪N!, and given W and X, we can effectively find A, B, . . , N.*

The proof that Theorem A follows from Theorem B given in [1] still holds, since it did not depend on the particular connectives present. Hence, we need only prove Theorem B in the more general case. In the course of this proof, we will use some notations that we used in the proof of the corresponding proof in [1]. In addition, the following new notations will be necessary. If *c* is a singulary connective, and *cQ* is a well-formed formula, then we will call *Q* the antecedent of *cQ* and will write *Q* = *AcQ*. Similarly, if *c* is a binary connective and *PcQ* is a well-formed formula, we will call *P* the antecedent of that formula and *Q* the consequent of that formula and will write *P* = *APcQ* and *Q* = *CPcQ*. If *c* is of degree *n*, we will call its arguments the first argument, the second argument, etc. For example, if *c* is of degree three, *B* is the second argument in *c*(*A*,*B*,*C*). We define the *principal connective* of a well-formed formula which contains a connective as the first symbol appearing in the translation of that formula into Łukasiewicz notation. For the definitions of the other technical terms used in the proofs in this paper, the reader should refer to [1]. Many of the modifications needed in the proofs in this paper, when compared with the corresponding proofs in the first paper, appear in the lemma below.

Lemma 1. *Given two well-formed formulae W and X, one can effectively tell whether X is an* SS *instance of W; if X is such an instance, one may effectively assign to each variable of W the subformula of X which was substituted for it in W to get X.*

Proof: We do induction on the length of *W*. If *W* is of length one, it is either a variable or a constant. If *W* is a constant, *X* is not an SS instance unless it is that same constant. In that case, the constant corresponds to itself for the effective assignment. If *W* is a variable, then *X* is an SS instance of *W* and the whole formula *X* corresponds to the single variable *W*.

Now assume the lemma true for *W* of length *n*-1 or less, and suppose that *W* is of length *n*. Then, if *X* is of length *n*-1 or less, it is not an SS instance, since SS always maintains or increases length. If *X* is of length *n* or greater, we consider the principal connective of *W* and the principal connective of *X*. If these two differ, then *X* is not an SS instance of *W*. If they are the same, we consider separately the cases that the connective in question, say c, is of various degrees. If c is of degree one, X is an SS instance of W if and only if AX is an SS instance of AW. If the connective in question is of degree two, X is an SS instance of W if and only if AX is an SS instance of AW and CX is an SS instance of CW and the formulae assigned to variables occurring in both AW and CW by the two assignments (AX to AW and CX to CW) are the same, and similarly for ternary and higher connectives, in which cases the n-th argument of X must be an SS instance of the n-th argument of W and the formulae assigned to any variables occurring in more than one argument of W must be the same. The necessity of the conditions is immediate. Further, if the conditions are satisfied, we can list, for each variable in W, the formula to be substituted for it in W to obtain X by SS, so that X is an SS instance of W. But the lengths of AW and CW are both of necessity shorter than the length of W, so that the theorem follows by induction, the effective assignment desired for the variables of W being the union of the effective assignments for AW and CW, which union is consistent in the sense of assigning only one formula to each variable by the above discussion. Q.E.D.

The following lemma holds exactly as it did in the first paper.

Lemma 2. *To every formula W, there corresponds a unique generalized version $W\#$ of which W is an* SS *instance. Given W we can effectively find $W\#$ and can effectively establish the correspondence between the variables of $W\#$ and W through which W arises as an* SS *instance of $W\#$.*

The following lemma requires modifications in its proof because of the possible presence of new connectives.

Lemma 3. *Given two formulae W and X, each of which has the property that no variable occurs in it more than once, the intersection of $W!$ and $X!$ is either empty or representable as $A!$ for some formula A, and given W and X, we can effectively find A or show that the intersection is empty.*

Proof: We do induction of the length of the shorter of W and X, which, without loss of generality, we may assume to be W. If the length of W is one, the proof proceeds as before in the first paper. Now assume that the lemma is true in all cases when the shorter of the two formulae has length less than n, and assume that the length of W is n. We again consider the possibilities of principal connectives in W, taking the various degrees separately. If the principal connective of W is not the same as the principal connective of X, there can be no mutual SS instances. If the principal connectives are the same, we first suppose that this connective is of degree one; call the connective in question c. Then W is of the form cW' and X is of the form cX'. Hence, by the induction assumption, we can find a formula A' such that $A!$ is the intersection of $W!$ and $X!$ or show that this intersection is empty. If the intersection is empty, then so is the intersection for $W!$ and $X!$. Otherwise, we let our desired formula A be cA'. The case of binary and higher connectives follows as in the proof of the corresponding lemma in the first paper. Thus, we have a process which is effective at every step, either telling us that there is no instance in common or giving us a formula A such that the desired intersection is $A!$. Q.E.D.

The proof of Theorem B now follows through the use of the lemmas and corollaries of [6], since, given the proofs of the lemmas above, the rest of the proof of the corresponding theorem in [1] is independent of the connectives present. The proof of Theorem A then follows exactly as before except that, at each step, we need consider only those formulae which have $\supset$ as their principal connective as major premisses for MP. Hence, the results on the strong decidability of cut-logics hold for all partial propositional calculi with our deduction rules. This would seem to cover the majority of partial propositional calculi usually considered. In addition, the treatment of other deduction rules given in the paragraphs below for modal logics indicates that the presence of other rules, provided proper restriction is made on the number of times that they may be used, does not effect the strong decidability of calculi.

3. *Modal logics.* As far as axioms are concerned, the case of modal logics does not differ from that of propositional calculi, except for the presence of new connectives. But the proofs given in the last section of this paper showed that, for our purposes, the presence or absence of such connectives is not significant. As far as deduction rules are concerned, however, the situation is a little more complicated. For the sake of exposition, we will use the formulation found in Feys [2] for the calculi of the series S1 to S5. The first rule used in these systems is the rule S. Since we are not restricting the number of applications of the rule S, its presence is equivalent to that of the rule SS which we shall use instead, since it was used in the proofs above and simplifies the presentation. The rule of detachment for strong implication corresponds exactly to the rule modus ponens, except that a different connective is involved. Hence, our proofs above apply to this rule also. The rule of adjunction which states that from P and Q we are to conclude $P \& Q$ as a theorem is an example of the other types of rules that may be present in various systems. (Together with one other rule mentioned below, it is the sole such rule for the Feys' systems mentioned.) The final rule present is the rule of the replacement of strict equivalences, which says that if P is strictly equivalent to Q, then a theorem remains a theorem if Q is replaced everywhere in it by P. As considerations of elementary examples show that the unlimited application of either of these two new rules can give an infinite number of formulae as theorems, none of which is an SS instance of any other, we must modify our definition of *cut-logic* to allow a total of at most N applications of the rules other than SS and as many applications of SS as are desired. From this point on, we always use this modified sense of cut-logic.

Proposition 1. *The addition of the rule of adjunction to any strongly decidable cut-logic leaves that system strongly decidable.*

Proof: The proof of this proposition follows exactly like the proof of Theorem A, by induction on the number N. In the case that $N = 0$, there is no change in the proof. In the induction step, we must take into account that the new rule adjunction may be applied to give new theorems. As before the induction hypothesis tells us that $\langle M,R,N\rangle$ has the right structure, that there exists the appropriate finite set of formulae, say n of them which could serve as axiom schemata for $\langle M,R,N\rangle$ even if R were replaced by the empty set of rules. We call these formulae *the appropriate formulae*. Then the application of the rule adjunction gives us new formulae which may be the conjunction of any two of the n formulae available. Hence, the application of the rule adjunction gives us a possibility of the presence of n^2 new formulae, where as usual, there are no difficulties about the rule SS. Since the number of formulae available is, at any rate finite, the desired conclusion follows. Q.E.D.

Proposition 2. *The addition of the rule of replacement of strict equivalence to a strongly decidable cut-logic leaves that logic strongly decidable.*

Proof: The proof again follows by induction, using the observation that, since for every N, there exists the appropriate finite set of formulae (by the induction hypothesis), only a finite number of schemata can be available of the form "P is strictly equivalent to Q". Call these formulae $F_1, F_2, \ldots, F_n$, and the corresponding Q's $Q_1, Q_2, \ldots, Q_n$, and similarly for the P's. Also, since there are only a finite number of formulae available, they can have only a finite number of subformulae which are SS instances of one of the Q_i's; call these SS instances $Q'_1, Q'_2, \ldots, Q'_n$. Now consider all of the pairs (Q_i, Q'_j) where Q'_j is an SS instance of Q_i. Suppose that there are r such pairs. Let S_k be the substitution which carries the first element of the k-th pair into the second, and P'_k be the result of making the substitution S_k in P_i, where Q_i is the first element of the k-th pair. Note that there are only a finite number of such P'_k available. If P'_k resulted from substitution in P_i, call Q_i the Q *corresponding* to P'_k. Then, to each P'_k there is a unique Q corresponding, and, since there are only a finite number of formulae available, this Q can occur only a finite number of times in those formulae. Let T be the set of formulae arising from the appropriate set of formulae for $\langle M,R,N\rangle$ by substitution in one of its formulae for Q_i of the P'_k to which Q_i corresponds. The elements of T are the new formulae which are made available by the application of the rule of the replacement of strict equivalences one time. By the above observations, this set is finite. Again, closure under the rule SS is automatic. Hence, finiteness at level N will still give us finiteness at leven $N+1$, as desired. Q.E.D.

Theorem C. *Cut modal logics whose deduction rules are included in the collection: rule of substitution, rule of adjunction, rule of detachment for strong implication, and rule of replacement for strict equivalence are strongly decidable.*

Proof: This follows from Theorems A and B above and the propositions just proved. Q.E.D.

Since the modal logics covered above include the systems S1 through S5 and most of their variants, we may say, in general, that *cut modal logics are strongly decidable*.

4. *Conclusion.* In the study of recursiveness and decidability, it is

customary to note that, perhaps unfortunately, in general, even when one knows that there is a solution to a problem, one cannot find a recursive bound on the number of steps necessary for the achievement of this solution. (For example, see theorem XI on page 31 of [3] and the comments above it.) The results of [1] and this paper would seem to indicate that such inability to limit the number of steps in computation, at least in the case that the computations are proofs of theorems, is of a rather essential nature, in that, although when no restriction is placed on the lengths of proofs, it is not even the case that all of the systems available are decidable, yet when suitable limitations are placed on the lengths of proofs, a decidability of such a strong nature appears that, if axiom schemata were allowed, all proofs could be made of length one by a reaxiomatization which can be effectively carried out.

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