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GENERALIZED ORDINAL NOTATION

FREDERICK S. GASS

This paper* is a contribution to the theory of ordinal notation, and it should be accessible to those familiar with references [4], [5], and [8]. For a textbook reference, see §§11.7 and 11.8 of [7]. For standard results and notation of recursive function theory, we generally follow [7]. "System" always means ordinal notation system, and "number" means nonnegative integer.

The purpose of this paper is to explore a broad class of systems that generalize Kleene's notion of *r-system*. In a background section, following notation and terminology, we describe some prominent results from ordinal notation theory. Then in section 1 we review some facts about r-systems and describe the generalized systems. Since the mathematical notation can easily become forbidding in such investigations, we have adopted a simplified notation that is often dependent upon context for its full meaning. Section 2 pursues the study of three particular systems that are noteworthy for their resemblence to Kleene's S_1, including (in section 3) a maximality property. In section 4 we identify the segment of ordinals for which these systems provide notations. Since the systems are maximal for only a proper sub-class of the generalized systems, we turn our attention in section 5 to a result about the remaining generalized systems: as a class, they admit no maximal system.

Notation and Terminology The notation and terminology are roughly those of second-order recursive function theory. ϕ_e is the partial recursive function with Gödel number e, N is the set of nonnegative integers, and T is a fixed *T-predicate*. We assume familiarity with prenex normal forms involving the predicate T. For $m \in \{0, 1\}$ and $n \in N$, Σ_n^m and Π_n^m are the

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usual prefix classes of relations on N, including subsets of N as 1-ary relations. If, when in prenex normal form, the description of R has a matrix of the form "$T^{R_1, \ldots, R_i}(e, x_1, \ldots, x_k)$", we call e an *index* of the relation R. If the description is specifically in $\Sigma_n^m(\Pi_n^m)$ form, then e is a Σ_n^m-*index* (Π_n^m-*index*) of R. Δ_n^m is the class $\Sigma_n^m \cap \Pi_n^m$, and $e = 2^a \cdot 3^b$ is a Δ_n^m-*index* of R if and only if a is a Σ_n^m-index of R and b is a Π_n^m-index of R. If $e = 2^a \cdot 3^b$, then $(e)_0 = a$ and $(e)_1 = b$.

Ord is the class of all ordinal numbers, and II is the class of countable ordinals. An ordinal α is classified as Σ_n^m (Δ_n^m, Π_n^m) if and only if that prefix class contains a well-ordering of order type α. We take well-orderings to be reflexive.

If R is a k-ary relation, we write "$R(x_1, \ldots, x_k)$" as well as "$(x_1, \ldots, x_k) \in R$". The *field* of a binary relation R is $\{x : \exists y[R(x,y) \vee R(y,x)]\}$. If S is a well-ordering with x and y in its field, then x *S-precedes* y if and only if $S(x, y)$ and $x \neq y$.

Let F and G be functions, each defined on some subset of N^k, with values in N. Then $F(x_1, \ldots, x_k) \simeq G(x_1, \ldots, x_k)$ if and only if either both sides are undefined, or else both are defined with the same value. If $E(x_1, \ldots, x_k)$ is a mathematical expression having at most one numerical value for each k-tuple of numbers, then $\lambda x_1 \ldots x_k[E(x_1, \ldots, x_k)]$ is the function F defined by $F(x_1, \ldots, x_k) \simeq E(x_1, \ldots, x_k)$.

We use the Recursion Theorem in this form:

If F is a $(k+1)$-ary partial recursive function, then there is a number c such that $\phi_c = \lambda x_1 \ldots x_k[F(c, x_1, \ldots, x_k)]$

An *ordinal notation system* is a pair $(A, |\ |)$ with $A \subseteq N$ and $|\ |$ a function from A into *Ord*. A is the set of *names* for ordinals, and $|\ |$ is the *naming function*. $|A| = \{|x| : x \in A\}$ is the set of ordinals "named by $(A, |\ |)$." If $\mathcal{S}$ is a collection of systems, then the system $(A, |\ |)$ is a *maximal system in* $\mathcal{S}$ if and only if $(A, |\ |) \in \mathcal{S}$, and $|B| \subseteq |A|$ for every $(B, |\ |) \in \mathcal{S}$.

As quantified variables, we use Latin lowercase letters to range over N, and Latin capitals to range over the class of relations on N.

Background In [8] Spector proved the surprising fact that every Δ_1^1 ordinal is in fact a Δ_1^0 ordinal, which is to say, recursive. Spector's actual result is that $\omega_1^A = \omega_1$ for $A \in \Delta_1^1$, a statement equivalent to the one mentioned here. Later Kreisel observed that even the Σ_1^1 ordinals are recursive. To display the known equalities and proper inclusions among prefix classes of ordinals, we offer the following diagram. We note that each class of ordinals is a proper initial segment of II.

(1)
$$\begin{tikzcd}[column sep=small, row sep=small]
 & & \Sigma_1^1 \arrow[dr, phantom, sloped, "\subset"] & & \Sigma_2^1 \arrow[dr, phantom, sloped, "\subset"] & \\
\Delta_1^0 = \Delta_1^1 \arrow[urr, phantom, sloped, "="] \arrow[drr, phantom, sloped, "\subset"] & & & \Delta_2^1 \arrow[ur, phantom, sloped, "\subset"] \arrow[dr, phantom, sloped, "="] & & \Delta_3^1 \\
 & & \Pi_1^1 \arrow[ur, phantom, sloped, "\subset"] & & \Pi_2^1 &
\end{tikzcd}$$

Beyond the class of Δ_3^1 ordinals, the equalities and inclusions depend upon which set theoretic axioms one assumes. See [1] and [2] for consequences of the axiom of constructibility and the axiom of determinateness. In [1], assuming the axiom of constructibility, Addison proves the Basis Theorem for Σ_n^1, which is crucial to the proof of

$$\begin{array}{ccc} & & \Sigma_n^1 \\ & \nearrow_{\subsetneq} & \\ \Delta_n^1 & & \\ & \searrow_{\subsetneq} & \\ & & \Pi_n^1 \end{array} \qquad \text{for } n \geq 3.$$

1. *R-systems and generalized r-systems* In [4] an r-system is defined to be a system $(A, |\ |)$ satisfying the following conditions: there are partial recursive functions ϕ_k, ϕ_p, and ϕ_q such that

(i) If $|x| = 0$, then $\phi_k(x) = 0$;
(ii) If $|x| = \alpha + 1$, then $\phi_k(x) = 1$ and $|\phi_p(x)| = \alpha$;
(iii) If $|x|$ is a limit ordinal α, then $\phi_k(x) = 2$ and $|\phi_q(x, 0)|$, $|\phi_q(x, 1)|$, $|\phi_q(x, 2)|$, . . . is a fundamental sequence for α.

One easily proves that $|A|$ is a proper initial segment of Π. An ordinal is *constructive* if and only if it is named by an r-system, and Π_c is the class of constructive ordinals, By results of Markwald and Spector, we may add "$\Pi_c = \Delta_1^0$" to the left end of diagram (1): an ordinal is constructive if and only if it is recursive.

The existence of an r-system that names all the constructive ordinals was established already in [4], where Kleene described three such systems. We describe here the first of the three, $S_1 = (\hat{A}, |\ |)$. For each ordinal α, let N_α be the set of names in $\hat{A}$ for α. Then

$N_0 = \{1\}$;
$N_{\alpha+1} = \{2^x : x \in N_\alpha\}$;
for limit ordinal α, $N_\alpha = \{3 \cdot 5^e :$ *ϕ_e is a recursive function and* $|\phi_e(0)|$, $|\phi_e(1)|$, $|\phi_e(2)|$, . . . *is a fundamental sequence for* $\alpha\}$.

Here we follow [7], p. 207, which is a nonessential modification of Kleene's original definition of S_1. Having defined N_α, we then set $\hat{A} = \bigcup_\alpha N_\alpha$ and $|x| = \alpha$ if and only if $x \in N_\alpha$.

Before presenting the generalized r-systems, let us recall that the entire preceding discussion can be "relativized" to any given relation R. One simply uses everywhere in place of partial recursive functions ϕ_a, functions ϕ_a^R partial recursive in R. The definitions and notation are also relativized; for example, "r^R-system", "Π_c^R", and "$S_1^R = (\hat{A}^R, |\ |)$".

Generalized r-systems In addition to straightforward relativization, we can generalize the notion of r-system by using various prefix classes of relations in place of the partial recursive functions. A preliminary definition will make our description of the generalized systems easier.

Definition 1. Let R be a $(k+1)$-ary relation on N. If $(x_1, \ldots, x_k, a) \in N^{k+1}$, then $!R(x_1, \ldots, x_k, a)$ if and only if a is the unique x such that $R(x_1, \ldots, x_k, x)$.

In order to avoid making three very similar definitions, we shall use "C" to represent the various prefix classes. A *generalized r-system* is any C-system, where "*C-system*" is defined as follows.

Definition 2. Let C be Σ_n^m, Δ_n^m, or Π_n^m for fixed m and n. An ordinal notation system $(A, |\ |)$ is a C-system if and only if there are relations K, P, and Q in C such that

(i) If $|x| = 0$, then $!K(x, 0)$;
(ii) If $|x| = \alpha + 1$, then $!K(x, 1)$ and $|p| = \alpha$, where $!P(x, p)$;
(iii) If $|x|$ is a limit ordinal α, then $!K(x, 2)$ and $|s_0|$, $|s_1|$, $|s_2|$, . . . is a fundamental sequence for α, where $!Q(x, n, s_n)$ for each $n \in N$.

Thus the *auxiliary relations* K, P, and Q enable one to recognize and deal with the various kinds of ordinals, via their names in A. As with r-systems, the set $|A|$ is a proper initial segment of II. Obviously every r-system is also a generalized r-system, being a Σ_1^0-system; and if C and D are two prefix classes, C a subclass of D, then every C-system is a D-system.

Let us call a C-system *arithmetical* if C is one of the artihmetical prefix classes. We shall show in section 4 that the Δ_1^1-systems as a class name precisely the Δ_1^1-ordinals, and as a result constitute no ordinal-naming improvement over the r-systems (recall "$II_c = \Delta_1^0$" and diagram (1)). Consequently we shall disregard entirely the arithmetical case, concentrating on the C-systems with $C = \Sigma_n^1$, Δ_n^1, or Π_n^1, for $n \geq 1$.

2. *Generalized r-systems similar to* S_1 Letting C be Σ_n^1, Δ_n^1, or Π_n^1 for some fixed $n \geq 1$, we can describe a C-system that bears to other C-systems a relationship similar to the one between S_1 and other r-systems.

Definition 3. The system $(\hat{C}, |\ |)$ is defined as follows. For each ordinal α, let N_α be the set of names in $\hat{C}$ for α. Then $\hat{C} = \bigcup_\alpha N_\alpha$, where

$N_0 = \{1\}$;
$N_{\alpha+1} = \{2^x : x \in N_\alpha\}$;
for limit ordinal α, $N_\alpha = \{3 \cdot 5^e :$ *e is an index for a function F in the prefix class C such that $|F(0)|$, $|F(1)|$, $|F(2)|$, . . . is a fundamental sequence for α*$\}$.

To verify that $(\hat{\Sigma}_n^1, |\ |)$, $(\hat{\Delta}_n^1, |\ |)$, and $(\hat{\Pi}_n^1, |\ |)$ are C-systems for appropriate C, let us consider possible auxiliary relations for them. Clearly there are recursive functions able to play the roles of K and P (in fact, the same K and P for all three systems), but the relation Q is necessarily more complex. In the case of $(\hat{\Sigma}_n^1, |\ |)$ and $(\hat{\Delta}_n^1, |\ |)$, Q may be taken to be Σ_n^1; in the case of $(\hat{\Pi}_n^1, |\ |)$ and $(\hat{\Delta}_n^1, |\ |)$ Q may be taken to be Π_n^1. Thus $(\hat{\Delta}_n^1, |\ |)$ is both a Σ_n^1-system and a Π_n^1-system.

Specifically, consider $(\hat{\Sigma}_2^1, |\ |)$. We take

$$K(x,y) \text{ if and only if } [x = 1 \;\&\; y = 0] \vee [2 \mid x \;\&\; y = 1] \vee [\textit{otherwise } y = 2]$$
$$P(x,y) \text{ if and only if } [x \textit{ has the form } 2^n \;\&\; y = n] \vee [\textit{otherwise } y = 1]$$

(2) $\quad Q(x, n, y)$ if and only if $[x$ *has the form* $3\cdot 5^e$ & $\exists A \forall B \exists w T^{A,B}(e, x, n, y, w)] \vee [$*otherwise* $y = 1]$.

Now if $|x|$ is a limit ordinal α, then $x = 3\cdot 5^e$ where e is a Σ_2^1-index of a function F whose successive values name a fundamental sequence for α. Hence $!Q(x, n, F(n))$ for every n; and so Q is a suitable auxiliary function for $(\hat{\Sigma}_2^1, |\ |)$. By the form of (2), we observe that $(\hat{\Sigma}_2^1, |\ |)$ is in fact a Σ_2^1-system. To show that $(\Delta_2^1, |\ |)$ is a Σ_2^1-system, we simply define the third auxiliary function by

$$Q(x, n, y) \text{ if and only if } [x \textit{ has the form } 3\cdot 5^e \;\&\; \exists A \forall B \exists w T^{A,B}((e)_0, x, n, y, w)] \vee [\textit{otherwise } y = 1].$$

Is $(\hat{\Delta}_2^1, |\ |)$ a Δ_2^1-system? A by-product of section 5 will be the negative answer to this question. For now, let us show that the three systems mentioned above provide names for the same segment of ordinals.

Proposition 1. $|\hat{\Sigma}_n^1| = |\hat{\Delta}_n^1| = |\hat{\Pi}_n^1|$, *for* $n \geq 1$.

Proof: The proof is based upon the fact that every total Σ_n^1 function F is a total Π_n^1-function, and vice versa. In fact, one can obtain a Π_n^1-index (Σ_n^1-index) for F uniformly and effectively from a given Σ_n^1-index (Π_n^1-index) e. To substantiate this well-known fact, one observes that

(3) $\quad F(x) = y$ if and only if $\forall z[F(x) = z \rightarrow z = y]$.

A Σ_n^1 description of "$F(x) = z$" gives rise to a Π_n^1 description of "$F(x) = y$", and likewise with the prefix classes interchanged.

We shall prove only $|\hat{\Sigma}_n^1| \subseteq |\hat{\Pi}_n^1|$, since the other inclusions are proved similarly. If F is a total Σ_n^1 function with index e and ϕ_c is a recursive function, one can evidently obtain a Π_n^1-index for $\phi_c F$ uniformly and effectively from e and c. Let h be a binary recursive function that accomplishes this task. Now consider the partial recursive function

(4)
$$\phi_c(y) \simeq \begin{cases} 1, & \text{if } y = 1 \\ 2^{\phi_c(x)}, & \text{if } y = 2^x \neq 1 \\ 3\cdot 5^{h(e,c)}, & \text{if } y = 3\cdot 5^e \\ 0, & \text{otherwise} \end{cases}$$

With c regarded as a variable, the right side of (4) is a partial recursive function of two variables. By the Recursion Theorem, then, (4) is a legitimate definition of ϕ_c. One easily proves by mathematical induction that ϕ_c is in fact recursive, being total. Next one proves by transfinite induction on $|y|$ that $y \in \hat{\Sigma}_n^1$ implies

(5) $\quad \phi_c(y) \in \hat{\Pi}_n^1$ and $|\phi_c(y)| = |y|$.

If $|y| = 0$, then $y = 1$ and (5) is obvious. If $|y| = \alpha + 1$, then y has the form 2^x, $|x| = \alpha$, then (5) follows by the inductive hypothesis. If $|y|$ is a limit ordinal α, then y has the form $3\cdot 5^e$ where e is a Σ_2^1-index of a total function F whose successive values name a functional sequence for α. By properties of h and the Gödel number c, and by the inductive hypothesis, we again conclude (5). Q.E.D.

3. *Maximality of* $(\hat{C}, |\ |)$ Having compared the systems $(\hat{\Sigma}^1_n, |\ |)$, $(\hat{\Delta}^1_n, |\ |)$, and $(\hat{\Pi}^1_n, |\ |)$ with each other, we now turn to the maximality properties that show how they compare with other C-systems. A useful lemma is this second-order version of the Recursion Theorem.

Lemma *Let C be* Σ^1_n or Π^1_n, for $n \geq 1$. *If S is a $(k+1)$-ary relation in C, then there is a number c such that the relation R with C-index c satisfies $R(x_1, \ldots, x_k)$ if and only if $S(c, x_1, \ldots, x_k)$.*

Proof: The form of this proof depends upon the specific T-predicate one adopts. Suppose, for example, that C is Σ^1_2, k is 1, and "$T^{A,B}(e, \ldots, w)$" means that $\phi^{A,B}_e(\ldots)$ converges by the w^{th} step of computation. Let S have Σ^1_2-index e, and let c be the number such that $\phi^{A,B}_c = \lambda x_1[\phi^{A,B}_e(c, x_1)]$, as provided by the relativized Recursion Theorem. If R is the Σ^1_2 relation with index c, then $R(x_1)$ if and only if $\exists A \forall B \exists w T^{A,B}(c, x_1, w)$ if and only if $\exists A \forall B \exists w T^{A,B}(e, c, x_1, w)$ if and only if $S(c, x_1)$. Q.E.D.

Now we are ready for the proposition of this section.

Proposition 2. $(\hat{\Sigma}^1_n, |\ |)$ *and* $(\hat{\Delta}^1_n, |\ |)$ *are maximal* Σ^1_n*-systems*; $(\hat{\Pi}^1_n, |\ |)$ *and* $(\hat{\Delta}^1_n, |\ |)$ *are maximal* Π^1_n*-systems, for* $n \geq 1$.

Proof: In light of Proposition 1, we may pass over $(\hat{\Delta}^1_n, |\ |)$. Since the proofs for the other two systems are entirely similar, we shall consider only $(\hat{\Sigma}^1_2, |\ |)$ as an example.

Let $(A, |\ |)$ be a Σ^1_2-system with auxiliary relations K, P, and Q. We shall define a new Σ^1_2 relation R such that for all $x \in A$,

$$(6) \qquad \exists y[y \in \Sigma^1_2 \ \& \ !R(x,y) \ \& \ |y| = |x|].$$

Delaying for a moment the justification for our definition, let R be defined by

$$(7) \qquad R(x,y) \text{ if and only if } \begin{cases} [K(x,0) \ \& \ y = 1] \vee \\ [K(x,1) \ \& \ \exists u \exists v[R(x,u) \ \& \ R(u,v) \ \& \ y = 2^v]] \vee \\ [K(x,2) \ \& \ y = 3 \cdot 5^{h(x)}], \end{cases}$$

where h is recursive, and $h(x)$ is a Σ^1_2-index of the relation F such that

$$(8) \qquad F(n,m) \text{ if and only if } \exists v[Q(x,n,v) \ \& \ R(v,m)].$$

Proceeding by transfinite induction on $|x|$, we prove that R satisfies (6) for all $x \in A$. If $|x| = 0$, then $!K(x,0)$, $!R(x,1)$, and (6) is true.

If $|x| = \alpha + 1$, then $!K(x,1)$ and $|p| = \alpha$, where $!P(x,p)$. By the inductive hypothesis, $\exists q[q \in \hat{\Sigma}^1_2 \ \& \ !R(p,q) \ \& \ |q| = |p|]$; and by the definition of R (second clause), $!R(x, 2^q)$. Thus (6) is satisfied by taking y to be 2^q.

If $|x|$ is a limit ordinal α, then $!K(x,2)$ and $|s_0|, |s_1|, |s_2|, \ldots$ is a fundamental sequence for α, where $!Q(x,n,s_n)$ for each n. By the choice of $h, h(x)$ is a Σ^1_2-index of a relation F such that $F(n,m)$ if and only if $R(s_n, m)$; and by the inductive hypothesis $F(n,m)$ implies $!F(n,m)$. So F is in fact a Σ^1_2-function with index $h(x)$ and the property that $|F(0)|, |F(1)|, |F(2)|, \ldots$ is a fundamental sequence for α. We conclude that $3 \cdot 5^{h(x)} \in \Sigma^1_2$ & $!R(x, 3 \cdot 5^{h(x)})$ & $|3 \cdot 5^{h(x)}| = |x|$. That is, (6) holds.

Finally, we must verify the existence of a Σ_2^1 relation R with the description (7). For the remainder of this proof, let "R_c" denote the Σ_2^1 relation with index c. We rephrase (7) and (8) as follows:

$$(7') \qquad R_c(x,y) \text{ if and only if } \begin{cases} [K(x,0) \ \& \ y = 1] \vee \\ [K(x,1) \ \& \ \exists u \exists v[P(x,u) \ \& \ R_c(u,v) \ \& \ y = 2^v]] \vee \\ [K(x,2) \ \& \ y = 3\cdot 5^{g(c,x)}], \end{cases}$$

where g is recursive, and $g(c,x)$ is a Σ_2^1-index of the relation F given by

$$(8') \qquad F(n,m) \text{ if and only if } \exists v[Q(x,n,v) \ \& \ R_c(v,m)].$$

The existence of such a recursive g is evident. Since the right side of (7′) is a ternary Σ_2^1 relation $S(c,x,y)$, we obtain the Σ_2^1 relation R_c for fixed c via the lemma. Thus the R of (7) is R_c, and the h of (7) is $\lambda x[g(c,x)]$. Q.E.D.

What we have proved is more than just the maximality of $(\hat{\Sigma}_2^1, |\ |)$. The relation R that enters into the proof establishes a kind of universality property much like the one Kleene established for his systems S_1 and S_3. R is a Σ_2^1 "liaison" between the given system $(A, |\ |)$ and $(\hat{\Sigma}_2^1, |\ |)$.

This is an appropriate place to indicate the motivation behind the definition we gave for "C-system". A more obvious generalization of the notion of r-system would be to require that K, P, and Q be functions, as are the ϕ_k, ϕ_p, and ϕ_q of an r-system. However, our primary objective was to find a generalization within which $(\hat{\Sigma}_n^1, |\ |)$, $(\hat{\Delta}_n^1, |\ |)$, and $(\hat{\Pi}_n^1, |\ |)$ would be C-systems for appropriate C, and such that the proof of their maximality would be fairly straightforward. The reader might ponder the difficulties that arise when K, P, and Q are required to be functions.

4. *The ordinals named by* $(\hat{C}, |\ |)$ By virtue of Proposition 1, we can study the classes $|\hat{C}|$ by considering $|\hat{\Sigma}_n^1|$ for $n \geq 1$. We show in this section that for fixed n this class of ordinals is exactly the class of Δ_n^1 ordinals. Our proof is in two parts, the first of which uses the fact that, for any relation R in Δ_n^1, Kleene's relativized r-system S_1^R is a Δ_n^1-system. Indeed, the auxiliary relations K and P for S_1^R might as well be the same ones described for $\hat{\Sigma}_2^1$ in section 2, and for Q we can take

$$Q(x,n,m) \text{ if and only if } x = 3\cdot 5^e \ \& \ \phi_e^K(n) = m.$$

The fact that Q is a Δ_n^1 relation follows from the recursive function theoretic result that $A \in \Delta_n^{1,R}$ & $R \in \Delta_n^1$ implies $A \in \Delta_n^1$ (c.f. [7], p. 412).

Proposition 3. *If α is a Δ_n^1 ordinal, then $\alpha \in |\hat{\Sigma}_n^1|$, for $n \geq 1$.*

Proof: Let R be a Δ_n^1 well-ordering of order type α. Since R is trivially recursive in R, α belongs to the class $\Delta_1^{0,R}$ of R-recursive ordinals. By relativized versions of results mentioned in section 1, $\alpha \in \Delta_1^{0,R}$ if and only if $\alpha \in II_c^R$ if and only if α is named by the maximal r^R-system S_1^R. Then, since S_1^R is known to be a Δ_n^1-system, hence a Σ_n^1-system, we conclude that α is named by the maximal Σ_n^1-system $(\hat{\Sigma}_n^1, |\ |)$. Q.E.D.

Proposition 4. *If $\alpha \in |\hat{\Sigma}_n^1|$, then α is a Δ_n^1 ordinal, for $n \geq 1$.*

Proof: We shall exhibit a recursive function ϕ_c with the property that for all $y \in \hat{\Sigma}^1_n$,

(9) $\phi_c(y)$ is an index of a Σ^1_n well-ordering of order type $\geq |y|$.

Before defining ϕ_c, we must describe some other functions that will enter into the definition of ϕ_c.

Given an index e of a Σ^1_n binary relation S, one can uniformly and effectively obtain an index of the Σ^1_n relation S' given by

$$S'(a, b) \text{ if and only if } [a > 0 \;\&\; b > 0 \;\&\; S(a-1, b-1)] \vee [b = 0 \;\&\; a > 0 \;\&\; S(a-1, a-1)].$$

If S is a well-ordering of order type α, then S' is a well-ordering of order type $\alpha + 1$. We take f to be a recursive function such that $f(e)$ is an index of S' whenever e is an index of S.

Suppose that for every $i \in N$, S_i is a Σ^1_n binary relation; and suppose that R is a Σ^1_n-binary relation with the property that for each $i \in N$, $\exists! e_i[R(i, e_i)$ and e_i is an index of $S_i]$. Then given an index of R, one can uniformly and effectively obtain an index of the Σ^1_n relation S' given by

$$S'(a, b) \text{ if and only if } \exists i \exists c \exists j \exists d\{a = 2^i \cdot 3^c \;\&\; b = 2^j \cdot 3^d \;\&\; [[i = j \;\&\; S_i(c, d)] \vee [i < j \;\&\; S_i(c, c) \;\&\; S_j(d, d)]]\}$$

If each of the S_i is a well-ordering, say of order type α_i, then S' is a well-ordering of order type $\Sigma_i \alpha_i$. We take g to be a recursive function such that $g(e)$ is an index of S' whenever e is an index of R.

The final preliminary function we need is a recursive function h with the property that $h(d, e)$ is a Σ^1_n-index of the "composition" of R and ϕ_d whenever e is a Σ^1_n-index of R. The composition S' is given by

$$S'(a, b) \text{ if and only if } \exists m[R(a, m) \;\&\; \phi_d(m) = b].$$

We let w_0 be a Σ^1_n-index of the empty well-ordering, and define ϕ_c via the Recursion Theorem as follows.

$$\phi_c(y) \simeq \begin{cases} w_0, & \text{if } y = 1 \\ f(\phi_c(x)), & \text{if } y = 2^x \neq 1 \\ gh(c, e), & \text{if } y = 3 \cdot 5^e \\ 0, & \text{otherwise}. \end{cases}$$

ϕ_c is easily seen to be recursive. One can prove that ϕ_c has property (9) for every $y \in \hat{\Sigma}^1_n$ by using transfinite induction on $|y|$. As usual, the cases to consider are

(a) $|y| = 0$, in which case $y = 1$

(b) $|y| = \alpha + 1$, in which case $y = 2^x$ and $|x| = \alpha$

(c) $|y|$ is a limit ordinal, in which case $y = 3 \cdot 5^e$ and e is a $\hat{\Sigma}^1_n$-index of a function F such that $|F(0)|$, $|F(1)|$, $|F(2)|$, . . . is a fundamental sequence for $|y|$.

We leave verification of these cases to the reader. Q.E.D.

Corollary *Let C be Σ_n^1, Δ_n^1, or Π_n^1 for $n \geq 1$. The system $(\hat{C}, |\ |)$ provides names for precisely the Δ_n^1 ordinals.*

5. *Maximal Δ_n^1-systems* Given that $(\hat{\Sigma}_n^1, |\ |)$ and $(\hat{\Pi}_n^1, |\ |)$ are maximal for Σ_n^1-systems and Π_n^1-systems, respectively, one might guess that $(\hat{\Delta}_n^1, |\ |)$ is maximal for Δ_n^1-systems. If it were in fact a Δ_n^1-system it would automatically be maximal by previous results and observations. In this section we shall answer the question about $(\hat{\Delta}_n^1, |\ |)$ by showing that there is no maximal Δ_n^1-system. The reader may recognize that our argument is an adaptation of one used by Putnam in [6]. A useful construction, described below, will enable us to focus on certain Δ_n^1-systems whose explicit description makes them more tenable than others.

Let $(A, |\ |)$ be a Δ_n^1-system with auxiliary relations K, P, and Q. If N_α is the set of names in A for the ordinal α, then the following relationships are implied by the definition of C-system.

$$N_0 \subseteq \{x: \ !K(x,0)\}$$

(10) $$N_{\alpha+1} \subseteq \{x: \ !K(x,1) \ \& \ \exists! p P(x,p)\}$$

For limit ordinals α, $N_\alpha \subseteq \{x: \ !K(x,2) \ \&$ *$\{(i,k): Q(x,i,k)\}$ is a total function whose successive values name a fundamental sequence for α*$\}$.

We shall keep inclusions (10) in mind while defining a new system $(B, |\ |)$ that names at least all of $|A|$.

We define B to be $\bigcup_\alpha M_\alpha$, where

$$M_0 = \{x: \ !K(x,0)\}$$

$$M_{\alpha+1} = \{x: \ !K(x,1) \ \& \ \exists p[!P(x,p) \ \& \ p \in M_\alpha]\}$$

For limit ordinals α, $M_\alpha = \{x$: *etc. as in* (10)$\}$.

The ordinal-naming function $|\ |$ is specified by $|\alpha|^{-1} = M_\alpha$ for each ordinal α. A little thought reveals that $(B, |\ |)$ is a Δ_n^1-system with K, P, and Q as auxiliary relations; and since $N_\alpha \subseteq M_\alpha$ for all α, we have $|A| \subseteq |B|$.

Proposition 5. *$|B|$ is a proper subclass of the class of Δ_n^1-ordinals.*

Proof: We shall describe a Δ_n^1 well-ordering R whose order type is exactly that of the set $|B|$. As a result, the order type cannot belong to the initial segment $|B|$.

If S is a well-ordering, we associate with each y in the field of S a unique ordinal $\bar{y}$ according to the rule $\bar{y}$ is the least ordinal greater than $\bar{x}$ for every x that S-precedes y. One proves by induction that the order type of S is precisely the least ordinal greater than $\bar{y}$ for every y in the field of S.

The well-ordering R promised above has the property that $\{\bar{x}$: x in the field of $R\} = |B|$. We define R by

$R(m,n)$ if and only if $\exists\alpha\exists\beta[\alpha < \beta$ & *m is the least integer in M_α* & *n is the least integer in M_β*$]$.

Observe that R can also be described as in (11) and (12) below, where we intend that G represent the set $\{(x,y): x \in M_{\bar{y}}\}$.

(11) $\exists S \exists G\{S \textit{ is a well-ordering} \;\&\; N \textit{ is the field of } S \;\&$
$\forall y[\{x: G(x,y)\} = M_{\bar{y}}] \;\&\; \exists x \exists y[x \textit{ S-precedes } y \;\&$
$G(m,x) \;\&\; G(n,y) \;\&\; \forall z[[G(z,x) \to m \leq z] \;\&\; [G(z,y) \to n \leq z]]]\}.$

(12) $\forall S \forall G\{[S \textit{ is a well-ordering} \;\&\; N \textit{ is the field of } S \;\&$
$\forall y[\{x: G(x,y)\} = M_{\bar{y}}] \;\&\; \exists y \forall x \neg G(x,y)] \to$
$\exists x \exists y[x \textit{ S-precedes } y \;\&\; \textit{etc. as in } (11)]\}.$

Our intention is to prove that (11) describes R as a Σ_n^1 relation, while (12) describes R as a Π_n^1 relation. Thus $R \in \Delta_n^1$. By inspection (recalling that "S is a well-ordering" is a $\Pi_1^{1,S}$ expression), the reader will see that it suffices to show the following expression to be in $\Sigma_n^{1,S,G}$ form:

(13) $$\{x: G(x,y)\} = M_{\bar{y}}$$

Assuming that S is a well-ordering with field N, we can express (13) as the conjunction of

(14) y is the first element of $S \to \{x: G(x,y)\} = \{x: !K(x,0)\}$

(15) $\forall z\,[y$ is the successor of z in $S \to \{x: G(x,y)\} =$
$\{x: !K(x,1) \;\&\; \exists p[!P(x,p) \;\&\; G(p,z)]\}]$

(16) y is a limit element in $S \to \{x: G(x,y)\} = \{x: !K(x,2) \;\&$
$\{(i,k): Q(x,i,k)\}$ *is a total function whose successive values name a fundamental sequence for* $\bar{y}\}$.

Bearing in mind that K, P, and Q are Δ_n^1 relations, we recognize that these expressions can all be put into $\Sigma_n^{1,S,G}$ prenex form, provided that the assertion about Q in (16) is not overly complex. We verify that this provision is satisfied, by expressing the assertion in detail:

$$\forall i \exists ! k Q(x,i,k) \;\&\; \forall i \forall j \forall k \forall l[[i < j \;\&\; Q(x,i,k) \;\&\; Q(x,j,l)] \to$$
$$\exists m \exists n[G(k,m) \;\&\; G(l,n) \;\&\; m \textit{ S-precedes } n \;\&\; n \textit{ S-precedes } y] \;\&$$
$$\forall v\,[v \textit{ S-precedes } y \to \exists i \exists k \exists m\,[Q(x,i,k) \;\&\; G(k,m) \;\&\; S(v,m)]]].$$

Thus the assertion is not overly complex, since it can plainly be put into $\Delta_n^{1,S,G}$ prenex form. Q.E.D.

Corollary *There is no maximal* Δ_n^1*-system for* $n \geq 1$.

In spite of this corollary, every Δ_n^1 ordinal is named by some Δ_n^1-system. The proof of Proposition 3 shows that the order type of a Δ_n^1 well-ordering R is named by the Δ_n^1-system S_1^R.

REFERENCES

[1] Addison, J. W., "Some consequences of the axiom of constructibility," *Fundamenta Mathematicae*, vol. 46 (1959), pp. 337-357.

[2] Addison, J. W., and Y. Moschovakis, "Some consequences of the axiom of definable determinateness," *Proceedings of the National Academy of Science, U.S.A.*, vol. 59 (1968), pp. 708-712.

[3] Gass, F. S., *The present state of ordinal notation theory*, Ph.D. thesis, Dartmouth College (1969).

[4] Kleene, S. C., "On notation for ordinal numbers," *The Journal of Symbolic Logic*, vol. 3 (1938), pp. 150-155.

[5] Kleene, S. C., "Hierarchies of number-theoretic predicates," *Bulletin of the American Mathematical Society*, vol. 61 (1955), pp. 193-213.

[6] Putnam, H., "On hierarchies and systems of notation," *Proceedings of the American Mathematical Society*, vol. 5 (1964), pp. 44-50.

[7] Rogers, H., Jr., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, New York (1967).

[8] Spector, C., "Recursive well-orderings," *The Journal of Symbolic Logic*, vol. 20 (1955), pp. 151-163.

Miami University
Oxford, Ohio