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AN UNSOLVABLE PROBLEM CONCERNING IDENTITIES

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In [1], Perkins has given a number of unsolvable problems concerning identities in varieties (equational classes) of algebras. In particular, he obtains from a semigroup with an unsolvable word problem, a variety of groupoids defined by a finite number of identities such that the problem of deciding whether an arbitrary identity holds in the variety is unsolvable. The following similar example may be of interest. It has the disadvantage compared with Perkins' example that the variety consists of algebras with two binary operations but to compensate for this the proof of undecidability is almost trivial.

Let $\mathfrak{S}$ be a semigroup with an unsolvable word problem, generated by a, b with defining relations $r_i(a,b) = s_i(a,b)$, $i = 1, 2, 3, \ldots$. Let $\mathfrak{F}$ be the absolutely free algebra (word algebra) on two binary operations $\oplus$, $\otimes$ and generators g_1, g_2. To a semigroup word $w = \alpha_1\alpha_2 \ldots \alpha_n$ where each α_i is a or b, we assign an $\mathfrak{F}$-word $\overline{w}(g_1,g_2)$ where $\overline{w}(x,y) = ((((x \circ_1 y) \circ_2 y) \circ_3 y) \ldots) \circ_n y$ and the operation $\circ_i$ is $\oplus$ or $\otimes$ according as α_i is a or b. For ease in reading, we will omit parentheses in products associated to the right so that the expression for $\overline{w}(x,y)$ above will be written as $x \circ_1 y \circ_2 y \circ_3 y \ldots \circ_n y$. We note that for u, v any semigroup words, $\overline{uv}(g_1,g_2) = \overline{v}(\overline{u}(g_1,g_2), g_2)$. Let $\mathfrak{V}$ be the variety of algebras with two binary operations and satisfying the identities $\overline{r}_i(x,y) = \overline{s}_i(x,y)$, for all x, y, $i = 1, 2, 3, \ldots$. $\mathfrak{V}$ is non-trivial since we can turn $\mathfrak{S}$ into a $\mathfrak{V}$-algebra by defining $u \oplus v = ua$, $u \otimes v = ub$, for all words u, v in $\mathfrak{S}$.

We recall that (i) $u = v$ in $\mathfrak{S}$ if we can transform u into v by a finite sequence of substitutions using the defining relations $r_i = s_i$ of $\mathfrak{S}$ and (ii) the identity $\overline{u}(x,y) = \overline{v}(x,y)$ holds in $\mathfrak{V}$ if we can transform $\overline{u}(g_1,g_2)$ in $\mathfrak{F}$ into $\overline{v}(g_1,g_2)$ by a finite sequence of substitutions using the defining identities $\overline{r}_i(x,y) = \overline{s}_i(x,y)$ of $\mathfrak{V}$.

(i) *If $u = v$ in $\mathfrak{S}$, then $\overline{u}(x,y) = \overline{v}(x,y)$ in $\mathfrak{V}$.*

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Proof If u is prq, v is psq, where $r = s$ is a defining relation of $\mathfrak{S}$, then $\overline{u}(g_1,g_2) = \overline{q}(\overline{r}(\overline{p}(g_1,g_2),g_2),g_2)$ and $\overline{v}(g_1,g_2) = \overline{q}(\overline{s}(\overline{p}(g_1,g_2),g_2),g_2)$. Hence, for v obtained from u by one application of the relations, the statement (i) is true. A simple induction on the number of applications of the relations needed to transform u into v completes the proof.

(ii) *If* $\overline{u}(x,y) = \overline{v}(z,y)$ *in* $\mathfrak{P}$, *then* $u = v$ *in* $\mathfrak{S}$

Proof Let $u = \alpha_1\alpha_2 \ldots \alpha_n$ where each α_i is a or b. Then $\overline{u}(g_1,g_2)$ is $g_1 \circ_1 g_2 \circ_2 g_2 \ldots \circ_n g_2$ where $\circ_i$ is $\oplus$ or $\otimes$ according as α_i is a or b. If $\overline{v}(g_1,g_2)$ is obtained from $\overline{u}(g_1,g_2)$ by one application of the defining identities, say substituting $\overline{s}(x,y)$ for $\overline{r}(x,y)$ in $\overline{u}(g_1,g_2)$, with appropriate $\mathfrak{F}$-words for x and y, then $\overline{r}(x,y)$ must be $x \circ_{i+1} y \circ_{i+2} y \ldots \circ_j y$, x must be $g_1 \circ_1 g_2 \circ_2 g_2 \ldots \circ_i g_2$ and y must be g_2. Writing $\overline{p}(g_1,g_2)$ for this value of x, we see that $\overline{u}(g_1,g_2)$ is $\overline{r}(\overline{p}(g_1,g_2),g_2) \circ_{j+1} g_2 \ldots \circ_n g_2$ and $\overline{v}(g_1,g_2)$ is $\overline{s}(\overline{p}(g_1,g_2),g_2) \circ_{j+1} g_2 \ldots \circ_n g_2$. Hence, u is $\alpha_1 \ldots \alpha_i r \alpha_{j+1} \ldots \alpha_n$, v is $\alpha_1 \ldots \alpha_i s \alpha_{i+1} \ldots \alpha_n$. That is, $u = v$ in $\mathfrak{S}$. Again, a simple induction on the number of substitutions necessary to transform $\overline{u}(g_1,g_2)$ into $\overline{v}(g_1,g_2)$ completes the proof.

It follows that the problem of deciding whether identities of the form $\overline{u}(x,y) = \overline{v}(x,y)$ hold in $\mathfrak{P}$ is unsolvable.

REFERENCE

[1] Perkins, Peter, "Unsolvable problems for equational theories," *Notre Dame Journal of Formal Logic*, vol. VIII (1967), pp. 175-185.

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