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AN EXTENSION OF NEGATIONLESS LOGIC

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§1. Nelson [1] has provided a formalization of part of Griss' negationless mathematics [2]. The logic Nelson devised uses a quantified implication $(A \supset\overline{x}\ B)$ and a quantified disjunction $(\Sigma\overline{x}(A_1, \ldots, A_n))$ as well as &, ∀, and ∃. These connectives do not exhaust the possibilities for rendering each provable sequent of Nelson's $\mathbf{P}_1$ system as a provable formula: when given a sequent $A_1, \ldots, A_m \rightarrow B_1, \ldots, B_n$, we lack a corresponding closed formula to be read negationlessly as "*for all* $x_1, \ldots, x_k$ *if* A_1 *and* . . . *and* A_m, *then* B_1 *or* . . . *or* B_n." Further, in Nelson's two most restricted predicate calculi there is no obvious way of forming Griss negation in several variables. If $\neq$ is a distinguishability relation and $P(t_1, \ldots, t_n)$ is a formula in which $x_1, \ldots, x_n$ do not occur, then the Griss negation of $P(t_1, \ldots, t_n)$ should be read "*for all* $x_1, \ldots, x_n$ *if* $P(x_1, \ldots, x_n)$ *then* $x_1 \neq t_1$ *or* . . . *or* $x_n \neq t_n$."

We have defined a general connective which provides the lacking notation [3]. Using the notation of [1] we give the definition and introduction rules for this connective. Let $\overline{x}$ be a non-empty list of distinct variables, Ψ a (possible empty) list of formulas, and Φ a non-empty list of formulas: then $(\Psi \supset\overline{x}\ \Phi)$ is a formula. Introduction rules suitable to P_2 - A_2 are

$$\frac{\Gamma, \Pi(\overline{x}) \rightarrow \Psi(\overline{x}) \mid \Gamma \rightarrow (\exists\overline{z})(\Pi(\overline{z})_1 \& \ldots \& \ \Pi(\overline{z})_m) \mid \Gamma \rightarrow (\exists\overline{z})\Psi(\overline{z})_1 \mid \ldots \mid \Gamma \rightarrow (\exists\overline{z})\Psi(\overline{z})_n}{\Gamma \rightarrow (\Pi(\overline{z}) \supset\overline{z}\ \Psi(\overline{z}))}$$

and

$$\frac{\Sigma \rightarrow A_1(\overline{t}), \Lambda \mid \ldots \mid \Sigma \rightarrow A_m(\overline{t}), \Lambda \mid B_1(\overline{t}), \Omega \rightarrow \Phi \mid \ldots \mid B_n(\overline{t}), \Omega \rightarrow \Phi}{(A_1(\overline{x}), \ldots, A_m(\overline{x}) \supset\overline{x}\ B_1(\overline{x}), \ldots, B_n(\overline{x})), \Sigma, \Omega \rightarrow \Lambda, \Phi}$$

in which Γ does not contain any of $\overline{x}$ free, each variable of $\overline{x}$ (term of $\overline{t}$) is free for the corresponding variable of $\overline{z}$ $(\overline{x})$ in each formula of $\Pi(\overline{z}), \Psi(\overline{z})$ $(A_1(\overline{x}), \ldots, B_n(\overline{x}))$, if $\Pi(\overline{x})$ $(A_1(\overline{t}), \ldots, A_m(\overline{t}))$ is empty then the premise(s) not involving $\Psi(\overline{x})$ $(B_1(\overline{t}), \ldots, B_n(\overline{t}))$ is (are) omitted, $\Pi(\overline{x})$ is a list of m formulas, $\Psi(\overline{x})$ is a non-empty list of n formulas, etc. An additional premise

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$$(A_1(\overline{x}), \ldots, A_m(\overline{x}) \supset \overline{x}\ B_1(\overline{x}), \ldots, B_m(\overline{x})),\ \Sigma,\ \Omega \rightarrow \mathcal{T}$$

is required for introduction in the antecedent in P_1 - A_1. The existence rule for both P_1 - A_1 and P_2 - A_2 is

$$\frac{\Gamma \rightarrow (A_1, \ldots, A_m \supset\overline{x}\ A_{m=1}, \ldots, A_n)}{\Gamma \rightarrow \exists\overline{x}A_i}$$

Let us denote by $P_i^!$ - $A_i^!$ the formal system which results from the following changes in P_i - A_i: the clauses in the definition of formula involving Σ, $\supset$, and $\forall$ are replaced by the clause for the general connective and the rules for Σ, $\supset$, and $\forall$ are replaced by the three rules for the general connective. (We abbreviate "general connective" by **gc**.) If F is a formula of P_i - A_i, let F' be defined inductively as follows: F' is F if F is prime; $(A \& B)'$ is $A' \& B'$; $(\exists xA)'$ is $\exists xA'$; $(\forall xA)'$ is $\supset x\ A'$; $(\Sigma\overline{x}(A_1, \ldots, A_n))'$ is $\supset\overline{x}\ A_1', \ldots, A_n'$; $(A \supset\overline{x}\ B)'$ is $A' \supset\overline{x}\ B'$. If Γ is a list of formulas then Γ' is the list of their maps; $(\Gamma \rightarrow \Phi)'$ is $\Gamma' \rightarrow \Phi'$.

Theorem I: *If* $S_1, \ldots, S_n \vdash S_{n+1}$ *in* P_i - A_i, *then* $S_1', \ldots, S_n' \vdash S_{n+1}'$ *in* $P_i^!$ - $A_i^!$.

The proof is an easy induction on the height of the deduction of S_{n+1} from $S_1, \ldots, S_n$.

§2. Before proving that $P_i^!$ - $A_i^!$ is a proper enlargement of P_i - A_i, we obtain a canonical form for formulas of P_2. Let $B, B_1, \ldots, B_m$ (A) contain free none of $x, \overline{x}$ $(\overline{y})$; let $\overline{u}, \overline{v}, \overline{w}$ be a list of distinct variables free only where exhibited; and let $\vdash A \longleftrightarrow B$ abbreviate $\vdash A \rightarrow B$ and $\vdash B \rightarrow A$.

Lemma I:

1. $\vdash \forall xB \longleftrightarrow B,$
2. $\vdash \exists xB \longleftrightarrow B,$
3. $\vdash (B_1 \supset\overline{x}\ B_2) \longleftrightarrow B_1 \& B_2,$
4. $\vdash (A \supset\overline{x}\ B) \longleftrightarrow \exists\overline{x}A \ \&\ B,$
5. $\vdash (B \supset\overline{x}\ C) \longleftrightarrow B \ \&\ \forall xC,$
6. $\vdash \Sigma\overline{x}(B_1, \ldots, B_m, C_1, \ldots, C_n) \longleftrightarrow B_1 \& \ldots \& B_m \& \exists\overline{x}C_1 \& \ldots \& \exists\overline{x}C_n,$
7. $\vdash \forall x(A \& B) \longleftrightarrow \forall xA \ \&\ B,$
8. $\vdash \exists x(A \& B) \longleftrightarrow \exists xA \&\ B,$
9. $\vdash (A \&\ B \supset\overline{x}\ C) \longleftrightarrow (A \supset\overline{x}\ C) \& B,$
10. $\vdash (A \supset\overline{x}C \&\ B) \longleftrightarrow (A \supset\overline{x}\ C) \&\ B,$
11. $\vdash (A \supset\overline{x}\overline{y}\ B) \longleftrightarrow \exists\overline{x}A \ \&\ \forall\overline{y}B,$
12. $\vdash \Sigma\overline{x}(A_1, \ldots, A_i \ \&\ B, \ldots, A_n) \longleftrightarrow \Sigma\overline{x}(A_1, \ldots, A_n) \&\ B,$
13. $\vdash P(\overline{u}) \supset\overline{u},\ Q(\overline{u}) \&\ R(\overline{u}) \longleftrightarrow (P(\overline{u}) \supset u\ R(\overline{u})) \& (P(u) \supset\overline{u}\ R(\overline{u})),$
14. $\vdash P(\overline{u}) \&\ Q(\overline{v}) \supset\overline{u}\overline{v}\overline{w}\ R(\overline{u}) \& S(\overline{w}) \longleftrightarrow [P(\overline{u}) \supset\overline{u}\ R(\overline{u})] \ \&\ \exists\overline{v}Q(\overline{v}) \& \forall\overline{w}S(\overline{w}).$

The results involving Σ and $\supset$ are easy consequences of the existence rules. To obtain the canonical form we use the Lemma to remove or reduce the depth of quantifiers wherever possible and to move & across quantifiers until we have a formula in which each formula (and formula of a conjunction) immediately within the scope of a quantifier has at least one of its free variables bound by the quantifier and in which no quantifier binds only one occurrence of a variable. Further, the premise and conclusion of each $\supset\overline{x}$ will have common free variables bound by the quantifier.

The canonical form for formulas composed entirely from one-place predicate letters is particularly simple: if F is such a formula with exactly $x_1, \ldots, x_m$ as free variables, then the canonical form of F is

$$Q_1 \& \ldots \& Q_n \& p_{11}(x_1) \& \ldots \& p_{1n_1}(x_1) \& \ldots \& p_{mn_m}(x_m)$$

where each Q_i is closed by the application of one quantifier containing exactly one variable to predicate letters and the conjunctions of distinct predicate letters and where x_i occurs free in F in exactly predicate letters $p_{i1}(x_i), \ldots, p_{in_i}(x_i)$.

While the only explicitly given propositional connective of Nelson's P_1 P_2 systems is conjunction, there remains the possibility that the quantifiers ($\forall$, $\exists$, $\supset$, and Σ) might be used to build up propositional connectives by using dummy variables; for example, one might ask if $\Sigma x(P,Q)$ is interpretable as "P or Q" when x does not occur free in P,Q. Lemma I shows that such constructions may always be reduced to conjunctions.

Remark: The reader may consult [4] for the development of the predicate calculus of P_2 including a replacement theorem.

§3. Let us denote by $\overline{\mathsf{P}}_2$ that system obtained by adding the **gc** and its rules to P_2; P_2 is then a subsystem of $\overline{\mathsf{P}}_2$. Let F be $p_1(x) \supset x\ p_2(x), p_3(x)$.

Theorem II: *There is no formula D of P_2 such that both $D \rightarrow F$ and $F \rightarrow D$ are provable in $\overline{\mathsf{P}}_2$.*

To prove this we regard the connectives as arithmetic truth functions. Let $\overline{x}, A, B, A(x), B(x), \ldots$ be evaluated by $\overline{\boldsymbol{x}}, \boldsymbol{\delta}, \boldsymbol{\sigma}, \boldsymbol{\delta}(\boldsymbol{x}), \boldsymbol{\delta}(\boldsymbol{x}), \ldots$; then $\forall x A(x)$ is evaluated by $\mathbf{sg}(\Sigma\boldsymbol{x}\boldsymbol{\delta}(\boldsymbol{x}))$; $\exists x A(x)$ by $\Pi\boldsymbol{x}\boldsymbol{\delta}(\boldsymbol{x})$; $A \& B$ by $\mathbf{sg}(\boldsymbol{\delta} + \boldsymbol{\sigma})$; $A(\overline{x}) \supset\overline{x}\ B(\overline{x})$ by $\mathbf{sg}[\{\Pi\overline{\boldsymbol{x}}\boldsymbol{\delta}(\overline{\boldsymbol{x}})\} + \{\Sigma\overline{\boldsymbol{x}}(\overline{\mathbf{sg}}(\boldsymbol{\delta}(\overline{\boldsymbol{x}})) \cdot \boldsymbol{\sigma}(\overline{\boldsymbol{x}}))\}]$; $\Sigma\overline{x}(A_1(\overline{x}), \ldots, A_m(\overline{x}))$ by $\mathbf{sg}(\{\Sigma\boldsymbol{i}(\Pi\overline{\boldsymbol{x}}\boldsymbol{\delta}_i(\overline{\boldsymbol{x}}))\} + \{\Sigma\overline{\boldsymbol{x}}(\Pi\boldsymbol{i}\boldsymbol{\delta}_i(\overline{\boldsymbol{x}}))\})$; $A_1(\overline{x}), \ldots, A_m(\overline{x}) \supset\overline{x}\ B_1(\overline{x}), \ldots, B_n(\overline{x})$ by $\mathbf{sg}(\{\Pi\overline{\boldsymbol{x}}(\Sigma\boldsymbol{i}\boldsymbol{\delta}_i(\overline{\boldsymbol{x}}))\} + \{\Sigma\boldsymbol{i}(\Pi\overline{\boldsymbol{x}}\boldsymbol{\sigma}_i(\overline{\boldsymbol{x}}))\} + \{\Sigma\overline{\boldsymbol{x}}[\overline{\mathbf{sg}}(\Sigma\boldsymbol{i}\boldsymbol{\delta}_i(\overline{\boldsymbol{x}})) \cdot \Pi\boldsymbol{i}\boldsymbol{\sigma}_i(\overline{\boldsymbol{x}})]\})$; $A \rightarrow B$ by $\boldsymbol{\sigma} \dot{-} \boldsymbol{\delta}$.

Also let the three truth functions $\boldsymbol{\alpha}(\boldsymbol{x})$, $\boldsymbol{\beta}(\boldsymbol{x})$, and $\boldsymbol{\gamma}(\boldsymbol{x})$ be defined on a domain of three objects by the following table:

	$\boldsymbol{\alpha}(\boldsymbol{x})$	$\boldsymbol{\beta}(\boldsymbol{x})$	$\boldsymbol{\gamma}(\boldsymbol{x})$
a	*0*	*0*	*1*
b	*0*	*1*	*0*
c	*1*	*1*	*1*

A straightforward induction argument shows that if $\vdash A \rightarrow B$ and $\vdash B \rightarrow A$ then A and B are identically equal. Note that there are but four cases out of twenty-seven in which F takes a value *1*. We will show that no closed formula of P_2 has such a small percentage of cases of value *1* (provided that it is not identically *0*).

First consider a formula D of $\mathbf{P}_2$ in one place predicate letters which is in the canonical form of §2. For D and F to be identically equal none of the Q_i's in D may be $\forall xE$ or $\Sigma x(E_1, \ldots, E_n)$ since all these closed formulas take only *1* as a value. Hence each Q_i must be of one of the following three forms:

a) $p_1(x) \& \ldots \& p_n(x) \supset x\ p_i(x)$ in which $i \leq n$;
b) $p_1(x) \& \ldots \& p_{n-1}(x) \supset x\ p_n(x)$ in which $p_n(x)$ is distinct from $p_i(x)$ for each $i < n$;
c) $\exists x(p_1(x) \& \ldots \& p_n(x))$.

If the formula in case c) is *1* for a particular assignment of α, β, γ to $p_1, \ldots, p_n$, then so are the formulas in cases a) and b). Let $r(n)$ be the ratio of cases in which $\exists x(p_1(x) \& \ldots \& p_n(x))$ takes value *1* to the total number of cases 3^n; than $r(n) = 1 - (2^n + 2^n - 1)/3^n$ since the formulas is *0* when only α, β or only α, γ are assigned. For $n \geq 2$, $r(n) \geq 6/27$ and for $n = 1$ both $\exists xp(x)$ and $p(x) \supset x\ p(x)$ are identically *0*. Thus no closed formula of P_2 in one place predicate letters is identically equal to F.

Next consider the evaluation of an arbitrary formula D in predicate letters $p_1(x_{11}, \ldots, x_{n_1}), \ldots, p_m(x_{m1}, \ldots, x_{mn_m})$ by a function δ. Since α, β, and γ are functions of one variable δ may be considered a function in prime arguments $\omega_1(x_1), \ldots, \omega_m(x_m)$. In obtaining a truth table for D we must consider all possible assignments of the variables x_i to some one of x_{ij} $(1 \leqq j \leqq n_i)$ as well as all possible assignments of α, β, γ to $\omega_1, \ldots, \omega_m$. If the assignment of variables is held fixed each variable of each Σx and Πx in δ is assigned to a variable of a quantifier of D, and a partition of the truth table into cases results. The evaluating function resulting form the fixed assignment is also an evaluating function for some formula in one place predicate letters; hence subdivision of the truth table has a ratio of cases *1* to total cases which is greater than *4/27*. Thus the ratio for D must also be greater than *4/27*.

§4. We take this opportunity to clarify the status of several of Nelson's existence rules. First, The independence of the existence rule 20a)

$$\frac{\Gamma \rightarrow A \supset \overline{x}\ B}{\Gamma \rightarrow \exists \overline{x} A}$$

can be established using the formula $(x = y' \supset y\ x = y') \supset x\ \exists yx = y'$ which is provable in $P_1 - A_1$. Following Nelson, let us map the formulas and sequents of $P_1 - A_1$ into a Gentzen type formal system for intuitionistic arthimetic by mapping $A \supset \overline{x}\ B$ into $\forall \overline{x}(A' \supset B')$ and $\Sigma \overline{x}(B_1, \ldots, B_m)$ into $\overline{x}(B'_1 \vee \ldots \vee B'_m)$. We find that the axioms and rules of inference except the existence rules map into theorems and derived rules of the intuitionistic system and that the above formula maps into $\forall x\ [\forall y(x = y' \supset x = y') \supset \exists xy = y']$. Were this formula provable then both $\exists yx = y'$ and $\exists y\ 0 = y'$ would be provable also. A similar argument shows the independence of 20b) and 21). The arguments remain valid for $P_2 - A_2$ and, for the quantified implication, for $P_3 - A_3$.

The existence rules 24a) and 24b) do not add to the theorems of P_1 - A_1. This may be established by proving the following lemma by induction on proofs.

Lemma II: a) *If* $\vdash A_0, \ldots, A_m \rightarrow \Phi$ *then* $\vdash \rightarrow \exists \bar{x}(A_{i_0} \& \ldots \& A_{i_k})$ *where* $i_0, \ldots, i_k$ *is a subset of* $0, \ldots, m$, $\exists \bar{x}(A_{i_0} \& \ldots \& A_{i_k})$ *is closed, and parentheses may be inserted in the conjunction in any way leading to a wff;* b) *If* $\vdash \Gamma \rightarrow B_1, \ldots, B_n$, *then* $\vdash \rightarrow \exists \bar{x} B_i$ *where* $\exists \bar{x} B_i$ *is closed.*

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