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EQUATIONAL CHARACTERIZATION OF NELSON ALGEBRA

DIANA BRIGNOLE

1. *INTRODUCTION.* H. Rasiowa in [8] and [9] has introduced the notion of **N**-lattice which plays a rôle in the study of the constructive logics with strong negation considered by David Nelson [7] and A Markov [4]. Not all axioms used by H. Rasiowa to characterize **N**-lattices, here called Nelson algebras, are equations. A paper published in collaboration with A. Monteiro, [3], gives a characterization of these algebras by equations but the proofs are heavily based on results indicated in [6] which have been obtained using transfinite induction. The purpose of this work, done under the guidance of Dr. A. Monteiro, is to indicate a purely arithmetical proof of that result. We reproduce here known results with the object of making this paper self-contained.

2. *THE DEFINITION OF H. RASIOWA.* Let us consider, in first place, the following definition;

2.1. *DEFINITION. A system* $\langle A, 1, \sim, \wedge, \vee\rangle$ *constituted by* 1°) *a non empty set* A, 2°) *an element* $1 \in A$ 3°) *a unary operator* $\sim$ *defined on* A, 4°) *two binary operations,* $\wedge$ *and* $\vee$, *defined on* A, *will be called a quasi-boolean algebra,* [1], *or a Morgan algebra,* [5], *if the following conditions are verified:*

N1. $x \vee 1 = 1$
N2. $x \wedge (x \vee y) = x$
N3. $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$
N4. $\sim\sim x = x$
N5. $(x \wedge y) = \sim x \vee \sim y$

A system $\langle A, \wedge, \vee\rangle$ verifying axioms N2 and N3 is, according to M. Scholander [10], a distributive lattice, from N1 we deduce that 1 is the last element of A. We can prove:

N'2. $a \vee (a \wedge b) = a$
N'3. $a \vee (b \wedge c) = (c \vee a) \wedge (c \vee b)$
N'5. $\sim(a \vee b) = \sim a \wedge \sim b$

and that $0 = \sim 1$ is the first element of A.

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We shall use the following properties of a distirbutive lattice with last element 1.

(α) $a \wedge a = a$
(β) $1 \wedge a = a$
(γ) $a \wedge (a \wedge b) = a \wedge b$
(δ) $a \vee (a \vee b) = a \vee b$

We shall write $a \leqslant b$ to indicate that $a = a \wedge b$. Let us consider now the definition of N-lattice introduced by H. Rasiowa in [8] and [9]:

2.2. *DEFINITION. A system* $\langle A, 1, \sim, \neg, \rightarrow, \wedge, \vee \rangle$ *constituted by* 1°) *a non empty set A.* 2°) *an element* $1 \in A$. 3°) *two unary operators:* $\sim, \neg$ *defined on A:* 4°) *three binary operations:* $\rightarrow, \wedge, \vee$ *defined on A, will be called a Nelson Algebra if the following axioms are verified:*

Axiom A1. (We write $a < b$ to indicate that $a \rightarrow b = 1$)

(1a). $a < a$

and

(1b). *if* $a < b$ *and* $b < c$ *then* $a < c$
Axioms A2. *The system* $\langle A, 1, \sim, \wedge, \vee \rangle$ *is a Morgan algebra, and on the other hand the relation* $\leqslant$ *defined by*

(E) $a \leqslant b$ *if and only if* $a < b$ *and* $\sim b < \sim a$

coincides with the order relation of the lattice $\langle A, \wedge, \vee \rangle$
Axiom A3. *If* $a < c$ *and* $b < c$ *then* $(a \vee b) < c$
Axiom A4. *If* $c < a$ *and* $c < b$ *then* $c < (a \wedge b)$
Axiom A5. $\sim(a \rightarrow b) < (a \wedge \sim b)$
Axiom A6. $(a \wedge \sim b) < \sim(a \rightarrow b)$
Axiom A7. $a < \sim\neg a$
Axiom A8. $\sim\neg a < a$
Axiom A9. $(a \wedge \sim a) < b$
Axiom A10. $a < (b \rightarrow c)$ *if and only if* $(a \wedge b) < c$
Axiom A11. $a = a \rightarrow 0$, *where* $0 = \sim 1$

In this definition, more than 11 axioms are really involved. Using the compact definition 2.1 of Morgan algebras the axiom A2 is equivalent to 6 axioms.

3. *THEOREM. If* $\langle A, 1, \sim, \neg, \wedge, \vee, \rightarrow \rangle$ *is a Nelson algebra then the following properties are verified:*

N1. $a \vee 1 = 1$
N2. $a \wedge (a \vee b) = a$
N3. $a \wedge (b \vee c) = (c \wedge a) \vee (b \wedge a)$
N4. $\sim\sim a = a$
N5. $\sim(a \wedge b) = \sim a \vee \sim b$
N6. $(a \wedge \sim a) \wedge (b \vee \sim b) = a \wedge \sim a$
N7. $a \rightarrow a = 1$

N8. $(a \to b) \wedge (\sim a \vee b) = \sim a \vee b$
N9. $a \wedge (a \to b) = a \wedge (\sim a \vee b)$
N10. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$
N11. $a \to (b \to c) = (a \wedge b) \to c$

PROOF: Properties N1-N5 are immediately verified since $\langle A, 1, \sim, \wedge, \vee \rangle$ is a Morgan algebra, according to axiom A2.

PROPERTY N6. $(a \wedge \sim a) \wedge (b \vee \sim b) = a \wedge \sim a$

This was established by H. Rasiowa [9], p. 79, whose proof we reproduce here:

Replacing b by $(b \vee \sim b)$ in axiom A9 we obtain

(1) $(a \wedge \sim a) \to (b \vee \sim b) = 1$

and using this result we can write

(2) $\sim (b \vee \sim b) \to \sim (a \wedge \sim a) = (b \wedge \sim b) \to (a \vee \sim a) = 1$

From (1) and (2) we obtain by axiom A2, $a \wedge \sim a \leqslant b \vee \sim b$.

PROPERTY N7. $a \to a = 1$

It follows from axiom A1.

PROPERTY N8. $(a \to b) \wedge (\sim a \vee b) = \sim a \vee b$

This formula has been established by H. Rasiowa in [9], 2.4 (d). From axioms A5 and A7 we obtain

(1) $\sim (a \to b) < a \wedge \sim b < \sim \neg a \wedge \sim b = \sim (\neg a \vee b)$

By axiom Al, $a \to 0 < a \to 0$, from which we obtain, by A10,

$$a \wedge (a \to 0) < 0, \qquad \text{i.e. } a \wedge \neg a < 0$$

As $0 < b$, by Al, $a \wedge \neg a < b$ from which we obtain, by A10,

(2) $\neg a < a \to b$

from $b \wedge a < b$ we obtain, applying again A10,

(3) $b < a \to b$

From (2), (3) and A3 it follows that

(4) $\neg a \vee b < a \to b$

From (1) and (4) we get, by A2,

(5) $\neg a \vee b \leqslant a \to b$

Now, we will prove that $\sim a \leqslant \neg a$. We have, by A9, $(\sim a \wedge a) < 0$, and then it follows, by A10,

(6) $\sim a < a \to 0 = \neg a$

Now, considering axiom A8

(7) $\sim\neg a < a = \sim\sim a$

From (6) and (7) we obtain, by A2,

(8) $\sim a \leqslant \neg a$

From (5) and (8) we finally obtain $\sim a \vee b \leqslant a \rightarrow b$

PROPERTY N9. $a \wedge (a \rightarrow b) = a \wedge (\sim a \vee b)$

This has been established in [3]. We shall now give a more direct proof. Making use of property N8 we have:

(1) $a \wedge (\sim a \vee b) \leqslant a \wedge (a \rightarrow b)$

We now proceed to prove that

(2) $a \wedge (a \rightarrow b) \leqslant a \wedge (\sim a \vee b)$

which is equivalent to the two following inequalities

(2a) $a \wedge (a \rightarrow b) < a \wedge (\sim a \vee b)$
(2b) $\sim(a \wedge (\sim a \vee b)) \rightarrow \sim(a \wedge (a \rightarrow b))$

By A1, $a \rightarrow b < a \rightarrow b$, by A10, $a \wedge (a \rightarrow b) < b$, and therefore

(3) $a \wedge (a \rightarrow b) < \sim a \vee b$

On the other hand $a \wedge (a \rightarrow b) < a$, then we get, from (3) and A4

(2a) $a \wedge (a \rightarrow b) < a \wedge (\sim a \vee b)$

From N3, N'3 and N4 we obtain

(4) $\sim(a \wedge (\sim a \vee b)) = \sim a \vee \sim(\sim a \vee b) = \sim a \vee (a \wedge \sim b)$

By axiom A6,

(5) $a \wedge \sim b < \sim(a \rightarrow b)$

From $a \wedge (a \rightarrow b) \leqslant a \rightarrow b$ we obtain

(6) $\sim(a \rightarrow b) < \sim(a \wedge (a \rightarrow b))$

From (5) and (6) we obtain

(7) $a \wedge \sim b < \sim(a \wedge (a \rightarrow b))$

From $a \wedge (a \rightarrow b) \leqslant a$ we obtain

(8) $\sim a < \sim(a \wedge (a \rightarrow b))$

Applying A3 to (7) and (8) we obtain

$$\sim a \vee (a \wedge \sim b) < \sim(a \wedge (a \rightarrow b))$$

and, therefore, by (4),

(2b) $\sim(a \wedge (\sim a \vee b)) < \sim(a \wedge (a \rightarrow b))$

which is what we wanted.

PROPERTY N10. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$

This formula has been established in [3].

(A) $\sim((a \to b) \wedge (a \to c)) < \sim(a \to (b \wedge c))$

Using A6, A2 and A5 we obtain

$$\sim(a \to b) < a \wedge \sim b \leqslant a \wedge \sim(b \wedge c) < \sim(a \to (b \wedge c))$$

Then, by A1, we can write

(1) $\sim(a \to b) < \sim(a \to (b \wedge c))$

Replacing b for c in (1) we obtain

(2) $\sim(a \to c) < \sim(a \to (b \wedge c))$

From (1) and (2), by axiom A3,

$$\sim((a \to b) \wedge (a \to c)) = \sim(a \to b) \vee \sim(a \to c) < \sim(a \to (b \wedge c))$$

(B) $\sim(a \to (b \wedge c)) < \sim((a \to b) \wedge (a \to c))$

By axiom A5, we have:

(1) $\sim(a \to (b \wedge c)) < a \wedge \sim(b \wedge c)$

By axiom A6, we can write

(2) $a \wedge \sim b < \sim(a \to b)$
(3) $a \wedge \sim c < \sim(a \to c)$

From (2) and (3) we obtain, using A3,

(4) $a \wedge (\sim b \vee \sim c) < \sim(a \to b) \vee \sim(a \to c)$

i.e.

(5) $a \wedge \sim(b \wedge c) < \sim((a \to b) \wedge (a \to c))$

From (1) and (5) we finally obtain

$$\sim(a \to (b \wedge c)) < \sim((a \to b) \wedge (a \to c))$$

(C) $a \to (b \wedge c) < (a \to b) \wedge (a \to c)$

In first place let us prove that:

(1) *if* $x < y$ *then* $a \to x < a \to y$

which is equivalent, by A10, to:

(1') *if* $x < y$, *then* $a \wedge (a \to x) < y$

From N9 and N3 we obtain

$$a \wedge (a \to x) = a \wedge (\sim a \vee x) = (a \wedge \sim a) \vee (a \wedge x)$$

By A9, $(a \wedge \sim a) < (a \wedge x)$, so we can write

$$a \wedge (a \to x) < a \wedge x < x$$

Then, if $x < y$: $a \wedge (a \to x) < y$. From $b \wedge c \leqslant b$ and $b \wedge c \leqslant c$ we obtain, using (1),

(2) $a \to (b \wedge c) < a \to b$
(3) $a \to (b \wedge c) < a \to c$

From (2), (3) and A4 we obtain

$$a \to (b \wedge c) < (a \to b) \wedge (a \to c)$$

(D) $(a \to b) \wedge (a \to c) < a \to (b \wedge c)$

By A1, $a \to b < a \to b$; then, by A10,

(1) $a \wedge (a \to b) < b$

In the same way:

(2) $a \wedge (a \to c) < c$

Applying A4 to (1) and (2) we obtain

$$a \wedge (a \to b) \wedge (a \to c) < b \wedge c$$

which is equivalent, by A10, to

$$(a \to b) \wedge (a \to c) < a \to (b \wedge c)$$

From (A), (B), (C) and (D) we obtain N10.

PROPERTY N11. $(a \wedge b) \to c = a \to (b \to c)$

This formula has been established by A. Monteiro [6], using transfinite induction. We give here an arithmetical proof:

(A) $(a \wedge b) \to c < a \to (b \to c)$

By axioms A1 and A10, we can write

$$\begin{aligned} 1 &= ((a \wedge b) \to c) \to ((a \wedge b) \to c) \\ &= (a \wedge b \wedge ((a \wedge b) \to c)) \to c \\ &= (a \wedge ((a \wedge b) \to c)) \to (b \to c) \\ &= ((a \wedge b) \to c) \to (a \to (b \to c)) \end{aligned}$$

which proves (A).

(B) $(a \to (b \to c)) < ((a \to b) \to c)$

By N8, $(\sim b \vee c) < b \to c$, then

(1) $a \wedge (\sim b \vee c) < b \to c$

By axiom A9, we can write

(2) $a \wedge \sim a < b \to c$

From (1) and (2) we obtain, applying A3,

$$a \wedge (\sim a \vee \sim b \vee c) = (a \wedge \sim a) \vee (a \wedge (\sim b \vee c)) < b \to c$$

Then, by axiom A10,

(3) $b \wedge a \wedge (\sim a \vee \sim b \vee c) < c$

From N3 and N9 we obtain:

$$\begin{aligned} b \wedge a \wedge (\sim a \vee \sim b \vee c) &= (b \wedge a \wedge \sim a) \vee (b \wedge a \wedge (\sim b \vee c)) \\ &= (b \wedge a \wedge \sim a) \vee (b \wedge a \wedge (b \rightarrow c)) \\ &= b \wedge a \wedge (\sim a \vee (b \rightarrow c)) \\ &= b \wedge a \wedge (a \rightarrow (b \rightarrow c)) \end{aligned}$$

which, using (3) gives:

$$b \wedge a \wedge (a \rightarrow (b \rightarrow c)) < c$$

Then, applying A10

$$a \rightarrow (b \rightarrow c) < (a \wedge b) \rightarrow c$$

(C) $\sim((a \wedge b) \rightarrow c) < \sim(a \rightarrow (b \rightarrow c))$

By axiom A1:

(1) $\sim((a \wedge b) \rightarrow c) < (a \wedge b) \wedge \sim c = a \wedge \sim(\sim b \vee c)$

and, by axiom A6:

(2) $a \wedge \sim(\sim b \vee c) < \sim(a \rightarrow (\sim b \vee c))$

From (1) and (2) we obtain

(3) $\sim((a \wedge b) \rightarrow c) < \sim(a \rightarrow (\sim b \vee c))$

Let us prove

(4) *If* $\sim x < \sim y$, *then* $\sim(z \rightarrow x) < \sim(z \rightarrow y)$

Surely, by A5, $\sim(z \rightarrow x) < z \wedge \sim x$; from the hypothesis we obtain: $z \wedge \sim x < z \wedge \sim y$. Then, we can write $\sim(z \rightarrow x) < z \wedge \sim y$. Besides, by axiom A5, $z \wedge \sim y < \sim(z \rightarrow y)$. So we can write $\sim(z \rightarrow x) < \sim(z \rightarrow y)$, and property (4) is proved. From axiom A6 and (4) we deduce:

(5) $\sim(a \rightarrow (\sim b \vee c)) < \sim(a \rightarrow (b \rightarrow c))$

(D) $\sim(a \rightarrow (b \rightarrow c)) < \sim((a \wedge b) \rightarrow c)$

By axiom A5, we have

(1) $\sim(a \rightarrow (b \rightarrow c)) < a \wedge \sim(b \rightarrow c)$

Also, by A5, we have: $\sim(b \rightarrow c) < b \wedge \sim c$, from which we obtain:

(2) $a \wedge \sim(b \rightarrow c) < a \wedge (b \wedge \sim c)$

From (1) and (2) we obtain:

(3) $\sim(a \rightarrow (b \rightarrow c)) < a \wedge b \wedge \sim c$

By axiom A5, we have:

(4) $(a \wedge b) \wedge \sim c < \sim((a \wedge b) \rightarrow c)$

From (3) and (4) we obtain

$$\sim(a \rightarrow (b \rightarrow c)) < \sim((a \wedge b) \rightarrow c)$$

From (A), (B), (C) and (D) we obtain property N11.

4. *THEOREM. Let* $\langle A, 1, \sim, \rightarrow, \wedge, \vee\rangle$ *be a system formed by* 1°) *a non empty set* A, 2°) *an element* $1 \in A$, 3°) *a unary operator* $\sim$ *defined on* A, 4°) *three binary operations*, $\rightarrow$, $\wedge$, $\vee$ *defined on* A, *and assume that properties* N1-N11 *are verified. If* $\neg x = x \rightarrow \sim 1$, *then the system* $\langle A, 1, \sim, \neg, \rightarrow, \wedge, \vee\rangle$ *is a Nelson Algebra.*

PROOF: Axiom A11 is verified by definition. The other axioms have to be proved. Let us first prove the two following lemmas:

4.1. *LEMMA. If* $a \leq b$ *then* $a \rightarrow b = 1$.

Let $a = a \wedge b$. Applying N7, N10 and (β) we obtain:

$$1 = a \rightarrow a = a \rightarrow (a \wedge b) = (a \rightarrow a) \wedge (a \rightarrow b) = 1 \wedge (a \rightarrow b) = a \rightarrow b$$

4.2. *LEMMA.* $a \rightarrow b = 1$ *if and only if* $a = a \wedge (\sim a \vee b)$.

(A) Assume that

(1) $a \rightarrow b = 1$

Then, applying (1) and N9, we obtain

$$a = a \wedge 1 = a \wedge (a \rightarrow b) = a \wedge (\sim a \vee b)$$

(B) Assume that

$$a = a \wedge (\sim a \vee b)$$

Applying N9 we obtain

$$a \wedge (a \rightarrow b) = a \wedge (\sim a \vee b) = a$$

i.e.: $a \leq a \rightarrow b$. Now by Lemma 4.1, N11 and (α)

$$1 = a \rightarrow (a \rightarrow b) = (a \wedge a) \rightarrow b = a \rightarrow b$$

Now, we shall prove axioms A1-A10 referred to, in definition 2.3.

Axiom A1. *If we write* $a < b$ *for* $a \rightarrow b = 1$, *we have* (1a) $a < a$, *and* (1b) *If* $a < b$ *and* $b < c$ *then* $a < c$.

(1a). It is an immediate consequence of N1
(1b). Let us consider $a < b$ and $b < c$, i.e. $a \rightarrow b = 1$ and $b \rightarrow c = 1$. By lemma 4.2 we can write:

(1) $a = a \wedge (\sim a \vee b)$
(2) $b = b \wedge (\sim b \vee c)$

From (1) we obtain, by N5 and N'5,

(3) $\sim a = \sim a \vee (a \wedge \sim b)$

Applying successively (1) and (2); N3, N2 and (3); N3 and N3; N'2 and N'2; N3, (2) and (1) we obtain:

$$\begin{aligned} a \wedge (\sim a \vee c) &= a \wedge (\sim a \vee b) \wedge (\sim a \vee (a \wedge \sim b) \vee c) \\ &= (\sim a \wedge a \wedge (\sim a \vee b)) \vee ((a \wedge \sim b) \wedge a \wedge (\sim a \vee b)) \vee \\ &\quad (c \wedge a \wedge (\sim a \vee b)) \\ &= (\sim a \wedge a) \vee ((a \wedge \sim b) \wedge (\sim a \vee b)) \vee ((c \wedge a) \wedge (\sim a \vee b)) \\ &= (\sim a \wedge a) \vee (b \wedge a \wedge \sim b) \vee (\sim a \wedge c \wedge a) \vee (b \wedge c \wedge a) \\ &= ((\sim a \wedge a) \vee (b \wedge a \wedge \sim b) \vee (b \wedge c \wedge a) \\ &= a \wedge (\sim a \vee (b \wedge \sim b) \vee (b \vee c)) \\ &= a \wedge (\sim a \vee (b \wedge (\sim b \vee c))) \\ &= a \wedge (\sim a \vee b) \\ &= a \end{aligned}$$

Then, by lemma 4.2, $a \rightarrow c = 1$, i.e. $a < c$.

Axiom A2. *The system* $\langle A, 1, \sim, \wedge, \vee \rangle$ *is a Morgan algebra, and* $a \leq b$ *is equivalent to* $a \rightarrow b = 1$ *and* $\sim b \rightarrow \sim a = 1$.

(A) It is immediate, from N1-N5 that the system is a Morgan algebra.

(B) *If* $a \leq b$ *then* $a \rightarrow b = 1$ *and* $\sim b \rightarrow \sim a = 1$.

Let us suppose $a \leq b$. Then by lemma 4.1 $a \rightarrow b = 1$. On the other hand, if $a \leq b$ then $\sim b \leq \sim a$. So, by lemma 4.1, we can write $\sim b \rightarrow \sim a = 1$.

(C) *If* $a \rightarrow b = 1$ *and* $\sim b \rightarrow \sim a = 1$, *then* $a \leq b$.

Let us suppose $a \rightarrow b = 1$ and $\sim b \rightarrow \sim a = 1$. By lemma 4.2 we have

(1) $a = a \wedge (\sim a \vee b)$
(2) $\sim b = \sim b \wedge (b \vee \sim a)$

Applying N3 to (1) and (2), we obtain:

(3) $a = (a \wedge \sim a) \vee (a \wedge b)$
(4) $\sim b = (\sim b \wedge b) \vee (\sim b \wedge \sim a)$

From N4, (4), N'5, N5 and N4 we obtain:

(5) $b = \sim\sim b = \sim(\sim b \wedge b) \wedge \sim(\sim b \wedge \sim a) = (\sim\sim b \vee \sim b) \wedge (\sim\sim b \vee \sim\sim a) =$
$(b \vee \sim b) \wedge (b \vee a)$

Applying successively (3) and (5); N3, N6, N2 and N2; N3, N3, (5) and (1), we obtain:

$$\begin{aligned} a \wedge b &= ((a \wedge \sim a) \vee (a \wedge b)) \wedge (b \vee \sim b) \wedge (b \vee a) \\ &= ((a \wedge \sim a) \wedge (b \vee \sim b) \wedge (b \vee a)) \vee ((a \wedge b) \wedge (b \vee \sim b) \wedge (b \vee a)) \\ &= ((a \wedge \sim a) \wedge (b \vee a)) \vee (a \wedge b) \\ &= (a \wedge \sim a) \vee (a \wedge b) \\ &= a \wedge (\sim a \vee b) \\ &= a \end{aligned}$$

Then, we can write $a \leq b$.

Axiom A3. *If* $a < c$ *and* $b < c$ *then* $a \vee b < c$.

A. Monteiro has proved that, in a Nelson algebra the equality $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$ holds: so that in particular, we have:

$$(a \to c) \wedge (b \to c) \leqslant (a \vee b) \to c$$

from which we immediately obtain axiom A3.

This equation was proved by A. Monteiro in the following way:

(A) *If* $x \leqslant y$ *then* $a \to x \leqslant a \to y$.

From $x = x \wedge y$ we obtain, applying N10,

$$a \to x = a \to (x \wedge y) = (a \to x) \wedge (a \to y)$$

i.e.: $a \to x \leqslant a \to y$.

(B) *If* $a \wedge x \leqslant \sim a \vee b$ *then* $x \leqslant a \to b$.

Let $a \wedge x \leqslant \sim a \vee b$. Then, by N9,

(1) $a \wedge x \leqslant a \wedge (\sim a \vee b) = a \wedge (a \to b) \leqslant a \to b$.

From (1) and (A) we obtain

(2) $a \to (a \wedge x) \leqslant a \to (a \to b)$

From

(3) $a \to (a \wedge x) = (a \to a) \wedge (a \to x) = 1 \wedge (a \to x) = a \to x$

and

(4) $a \to (a \to b) = (a \wedge a) \to b = a \to b$

we obtain $a \to x \leqslant a \to b$. From $x \leqslant a \to x$ and $a \to x \leqslant a \to b$, we have $x \leqslant a \to b$.

(C) $a \wedge (a \to c) \wedge (b \to c) \leqslant \sim b \vee c$

Applying successively N9, N3, N6, N3, N8, N3, N3 and N'2, and N'2 we obtain:

$$\begin{aligned} a \wedge (a \to c) \wedge (b \to c) &= a \wedge (\sim a \vee c) \wedge (b \to c) \\ &= (a \wedge \sim a \wedge (b \to c)) \vee (a \wedge c \wedge (b \to c)) \\ &= ((a \wedge \sim a) \wedge (b \to c)) \vee (a \to c) \\ &= ((b \vee \sim b) \wedge (b \to c)) \vee (a \wedge c) \\ &= (b \wedge (b \to c)) \vee (\sim b \wedge (b \to c)) \vee (a \wedge c) \\ &= (b \wedge (b \vee c)) \vee b \vee (a \wedge c) \\ &= (a \wedge c) \vee (b \wedge c) \vee (b \wedge \sim b) \vee \sim b \\ &= ((a \vee b) \wedge c) \vee \sim b \\ &= c \vee \sim b. \end{aligned}$$

(D) $a \wedge (a \to c) \wedge (b \to c) \leqslant \sim a \vee c$

Applying N9 we have $a \wedge (a \to c) \wedge (b \to c) = a \wedge (\sim a \vee c) \wedge (b \to c) \leqslant \sim a \vee c$

(E) $a \wedge (a \to c) \wedge (b \to c) \leqslant \sim (a \vee b) \vee c$

From (C) and (D) we have

$$a \wedge (a \to c) \wedge (b \to c) \leqslant (\sim a \vee c) \wedge (\sim b \vee c) = \sim (a \vee b) \wedge c$$

(F) $b \wedge (a \to c) \wedge (b \to c) \leqslant \sim(a \vee b) \vee c$

(F) is a consequence of E, replacing a by b.

(G) $(a \to c) \wedge (b \to c) \leqslant (a \wedge b) \to c$

From (E) and (F) we obtain

$$(a \vee b) \wedge (a \to c) \wedge (b \to c) \leqslant \sim(a \vee b) \vee c$$

Then, by (A)

$$(a \to c) \wedge (b \to c) \leqslant (a \vee b) \to c$$

Axiom A4. *If* $a < b$ *and* $a < c$ *then* $a < b \wedge c$.

Let $a < b$ and $a < c$, that is

(1) $a \to b = 1$,
(2) $a \to c = 1$

From N10, (1), (2), and (α) we have

$$a \to (b \wedge c) = (a \to b) \wedge (a \to c) = 1 \wedge 1 = 1$$

i.e.: $a < b \wedge c$.

Axiom A5. $\sim(a \to b) < a \wedge \sim b$.

By axiom N8, $\sim a \vee b \leqslant a \to b$, and therefore

(1) $\sim(a \to b) \leqslant \sim(\sim a \vee b) = a \wedge \sim b$

By lemma 4.4, we obtain from (1)

$$\sim(a \to b) \to (a \wedge \sim b) = 1$$

Axiom A6. $a \wedge \sim b < \sim(a \to b)$

We shall prove the equality

(1) $\sim(\sim a \vee b) \to \sim(a \to b) = 1$

i.e.:

(2) $(a \wedge \sim b) \to \sim(a \to b) = 1$

which is equivalent, by lemma 4.2 to

(3) $a \wedge \sim b = (a \wedge \sim b) \wedge (\sim(a \wedge \sim b) \vee \sim(a \to b))$

Applying $\sim$ to both members in (3) we have the equivalent equality:

(4) $\sim a \vee b = \sim a \vee b \vee (a \wedge \sim b \wedge (a \to b))$

which we can write, using N8,

(5) $\sim a \vee b = \sim a \vee b \vee (a \wedge (\sim a \vee b) \wedge \sim b)$

and since this equality is verified, the same occurs with (1)

Axiom A7. $a < \sim \neg a$.

This result is obtained replacing b by 0 in axiom A5 and observing that $\sim\sim a = a$ and $a \rightarrow 0 = \neg a$.

Axiom A8. $\sim\neg a < a$.

We obtain it replacing b by 0 in axiom A6.

Axiom A9. $a \wedge \sim a < b$.

By N8 $\sim a \vee b \leqslant a \rightarrow b$, and therefore $\sim a \leqslant a \rightarrow b$. Then, by lemma 4.1, we obtain.

(1) $\sim a \rightarrow (a \rightarrow b) = 1$

From (1) and N11 we obtain $(\sim a \wedge a) \rightarrow b = 1$, i.e.: $a \wedge \sim a < b$.

Axiom A10. *$a < b \rightarrow c$ is equivalent to $a \wedge b < c$.*

It is enough to observe that, by property N11, $a \rightarrow (b \rightarrow c) = 1$ is equivalent to $(a \wedge b) \rightarrow c = 1$. This ends our proof.

5. *CONCLUSION.* From theorem 3 and 4 we obtain a definition of Nelson algebra, which, *cf.* [3], is the following:

5.1. *DEFINITION. Let $\langle A, 1, \sim, \wedge, \vee, \rightarrow\rangle$ be a system constituted by* 1°) *a non-empty set A,* 2°) *an element $1 \varepsilon A$,* 3°) *a unary operator $\sim$ defined on A,* 4°) *three binary operations: $\wedge, \vee, \rightarrow$ defined on A. Such a system will be called a Nelson algebra if we define $\neg x = x \rightarrow \sim 1$, and if the following axioms are verified:*

N1. $a \vee 1 = a$
N2. $a \wedge (a \vee b) = a$
N3. $a \wedge (b \vee c) = (c \wedge a) \vee (b \wedge a)$
N4. $\sim\sim a = a$
N5. $\sim(a \wedge b) = \sim a \vee \sim b$
N6. $(a \wedge \sim a) \wedge (b \vee \sim b) = a \wedge \sim a$
N7. $a \rightarrow a = 1$
N8. $(a \rightarrow b) \wedge (\sim a \vee b) = \sim a \vee b$
N9. $a \wedge (a \rightarrow b) = a \wedge (\sim a \vee b)$
N10. $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$
N11. $a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c$

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Instituto de Matemâtica
Universidad Nacional del Sur
Bahia Blanca, Argentina