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PROPOSITIONAL CALCULUS IN IMPLICATION AND NON-EQUIVALENCE

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If we use *C* for implication, 0 for a false constant, and *J* for non-equivalence, $J\alpha\beta$ is definable as $CC\alpha\beta CC\beta\alpha 0$. Hence the full classical calculus in *C*-0-*J* is obtainable by substitution and detachment from

1. *CCCpqrCCrpCsp*
2. *C*0*p*
3. *CJpqCCpqCCqp*0
4. *CCCpqCCqp*0*Jpq*

Here 1 is Łukasiewicz's single axiom for *C*-pure; 2 with this is known to give full *C*-0, and 3 and 4 are jointly equivalent to the above definition. Moreover, we have *CJpp*0 from 3 *q*/*p* and *Cpp*, and *CJppq* from this and 2; from this in turn we have *CJppJqq*, showing that *Jpp* is a constant and can take the place of 0 in the above postulates to give a full set for *C*-*J*. 2 and 3 can then be replaced by *CJpqCCpqCCqpr*, which yields 3 by *r*/*Jpp*, and 2 by *q*/*p* and *Cpp*. Hence 1.*CCCpqrCCrpCsp*, 2'.*CJpqCCpqCCqpr* and 3'.*CCCpqCCqpJppJpq* suffice for *C*-*J*. This set somewhat abridges that given by Shukla in "A set of axioms for the propositional calculus with implication and non-equivalence", *Notre Dame Journal of Formal Logic*, Vol. 7 (1966), pp. 281-6.

Similar considerations show that if we use *B* for non-implication and axiomatise in *C*-*B* (as suggested by C. S. Peirce, *Collected Papers* 3.386), we need only 1, *CBpqCCpqr* and *CCCpqBppBpq*. Indeed, we can give a similar proof of an old result, the adequacy of 1, *CNpCpq* and *CCpNpNp* for *C*-*N*, thus:

5. *CNCppCCppq* (*CNpCpq*)
*6. *CNCppq* (1, 5)
*7. *CNCppNCqq* (6)
8. *CCpNCppCpq* (1, 6)
*9. *CCpNCppNp* (1, 8 *q*/*Np*, *CCpNpNp*)
*10. *CNpCpNCpp* (*CNpCpq*)

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Here 7 shows the constancy of *NCpp*, so that 6 can be read as *C*0*q* and 9 and 10 as defining *Np* as *Cp*0.

In each case (*C-J*, *C-B*, *C-N*) the added postulates are intuitionistically valid (with *J* for intuitionist *NE* and *B* for intuitionist *NC*) and the deductions go through if 1 is replaced by an axiom for *C*-positive. And we obtain exactly the same *C-J-B-N* theorems, given *C*-positive, from each of

(1) Above *C-J* pair, and Dff. $N\alpha = C\alpha J\alpha\alpha$, $B = NC$.
(2) Above *C-B* pair, and Dff. $N\alpha = C\alpha B\alpha\alpha$, $J\alpha\beta = CC\alpha\beta NC\beta\alpha$.
(3) Above *C-N* pair, and Dff. *B* as in (1), *J* as in (2).

Given, beyond these, an undefined *E* with axioms *CEpqCpq*, *CEpqCqp*, *CCpqCCqpEpq*, or given $E\alpha\beta$ as $KC\alpha\beta C\beta\alpha$ with the usual for *K*, we can prove *CJpqNEpq* and *CNEpqJpq*, and also *CEpqNJpq*, but not (from the intuitionist basis) *CNJpqEpq*—only *CNJpqNNEpq*. In fact, although *J* is (as intended) equivalent to *NE*, even intuitionistically, *E* has *no* equivalent in intuitionist *C-J*. This follows from its having none in intuitionist *C-N*, to which *C-J* is equivalent (intuitionistically as well as classically) in functional content.

Note: For the classical system, C. A. Meredith (letter of March 28, 1968) gives the following alternative axiomatisation in *C-B*:

1. *CCpqCCqrCpr*
2. *CpCCBpqqq*
3. *CqCBpqr*
4. *CBpqp*

In support of its sufficiency, he observes: "Let α be some thesis: define *Nq* as $B\alpha q$; then 2 gives *CCNqqq*, and 3 is *CqCNqr*, so with 1 we have *C-N*; now 2 gives *CpCNqBpq*, 3 gives *CBpqNq* and, from 4, *Bpq* = *KpNq* = *NCpq*." He also gives these independence proofs:

For (1):

C	1	2	0	
*1	1	1	0	(*Bpq*
2	1	1	1	= *CCpq*0).
0	1	1	1	

For (2): *Bpq* = Falsum; *C* normal.
For (3): *Bpq* = *p*; *C* normal.
For (4): *Bpq* = *Nq*; *C* normal.

And for the insufficiency of *CCpqCBpqr* as a replacement for 3 and 4, he gives

C	1	2	0	
*1	1	0	0	(*Bpq*
2	1	0	0	= *CCpq*0).
0	1	1	1	

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