266
Notre Dame Journal of Formal Logic
Volume X, Number 3, July 1969

EQUATIONAL POSTULATES FOR THE SHEFFER STROKE

C. A. MEREDITH

1. *Notation for equational reasoning.*[1] There are two fundamental rules of equational reasoning: (i) Euclid, i.e. $\alpha = \beta, \alpha = \gamma \rightarrow \beta = \gamma$; (ii) elaboration, i.e. $\alpha = b \rightarrow f\alpha = f\beta$ (and indeed $\alpha = \beta, \gamma = \delta \rightarrow g\alpha\gamma = g\beta\delta$), also given by Euclid in particular cases.

I number all formulae and deal only with constant terminal functors.

(i) I give as: if m and n are sets of equations, εmn is the set of equations $Q = R$ such that, for some P, $P = Q$ is in m and $P = R$ is in n.

(ii) I show by the insertion of '′' in the non-argument places of f and the insertion of (the number of) $\alpha = \beta$ in the argument places.

2. *Illustration and explanation.*[2] For example, suppose the equations (or more accurately, substitution classes of equations) numbered 1 and 2 are

1. $RRppRqp = p$
2. $RpRqRpr = RRRrqqp$

Then (a) the equation

$RpRRqqRpq = RRRqRqqRqqp$ is in 2, (since it is 2 q/Rqq, r/q),

and (b) the equation

$RpRRqqRpq = Rpq$ is in $R'1$,

since if we have $RRqqRpq = q$ (i.e. 1 p/q, q/p) for our $\alpha = \beta$, we could have $RpRRqqRpq$ for our $f\alpha$ (with f of the form R') and Rpq for our $f\beta$, and so the given equation for our $f\alpha = f\beta$. Further, given (a) and (b) we can infer that (c)

3. $RRRqRqqRqqp = Rpq$ is in $\varepsilon 2R'1$,

1. This notation is also used, in a sketchy way, in [1], Section 3.

2. This section is added by A. N. Prior.

Received May 12, 1968

for if we have the equations in (a) and (b) for our $P = Q$ and $P = R$, 3 will be our $Q = R$. And we may compress this whole proof to the line

3. $RRRqRqqRqqp = Rpq$ $\varepsilon 2R'1$.

Moreover, given this line we can reconstruct the proof. For if 3 is in $\varepsilon 2R'1$, the relevant members of 2 and $R'1$ must be of the forms

. . . . . . $= RRRqRqqRqqp$
. . . . . . $= Rpq$,

where both gaps are filled in by the same formula, and from 2 and the first line we can easily see what this formula must be. (Where alternative solutions are possible, we may choose the most general one which will give the same LHS on both sides, i.e. the one with fewest unnecessary identifications of variables).

The rule $\alpha = \beta,\ \gamma = \delta \rightarrow g\alpha\gamma = g\beta\delta$ can be proved from (i) and (ii) of the previous section, provided that we can prove $\alpha = \beta \rightarrow \beta = \alpha$; for we can proceed thus:

1. $\alpha = \beta$
2. $\gamma = \delta$
3. $g\alpha\gamma = g\beta\gamma$ $g1'$
4. $g\beta\gamma = g\alpha\gamma$ 3, converted
5. $g\beta\gamma = g\beta\delta$ $g'2$
6. $g\alpha\gamma = g\beta\delta$ $\varepsilon 45$

The symmetry of = is not in fact provable from (i) and (ii) alone, but it is provable when these are supplemented by the special axioms used in the examples below. (See end of next section). And in such cases it will be useful to refer to 6, in proof formulae, as $g12$. If 2 is a substitution in 1, 6 will of course be $g11$. Cases of this sort will occur below (e.g. R.29.29 in the proof of thesis 30 in the next section).

3. *First abridgement of Sheffer*. Using R either for joint or for alternative denial, the equational axioms

1. $RRppRqp = p$
2. $RRpRqrRpRqr = RRRrppRRqpp$

with the definition

3. $Rpp = Np$

will yield Sheffer's original equations for this functor. This result (of about 1949) is provable as follows (Sheffer's equations being starred):

4.	$RNpRqp = p$	$\varepsilon R3'1$	
5.	$RNpNp = p$	$\varepsilon R'34$	
*6.	$NNp = p$	$\varepsilon 35$	
7.	$p = p$	$\varepsilon 66$ (or $\varepsilon 11$)	
8.	$RRRrppRRqpp = NRpRqr$		$\varepsilon 23$
9.	$NRpRqq = NRRqpp$		$\varepsilon 83$

10.	$RRqpp = RpNq$	$\varepsilon\varepsilon\varepsilon N966R'3$
11.	$RRpNrRpNq = NRpRqr$	$\varepsilon R.10.10.8$
12.	$RNRqpp = Rqp$	$\varepsilon R'44$
13.	$RpNNRqp = RRqpp$	$\varepsilon 10.R.12.'$
14.	$RRqpp = RpRqp$	$\varepsilon 13.R'6$
15.	$RpRqp = RpNq$	$\varepsilon 14.10$
16.	$RpRqr = NRRpNrRpNq$	$\varepsilon 6\varepsilon N.11.7$
17.	$NRRNpNpRNpNq = p$	$\varepsilon.16.4$
18.	$NRpRNpNq = p$	$\varepsilon.NR5'.17$
19.	$NNRRpNNqRpNNp = p$	$\varepsilon.N.16.18$
20.	$NNRRpqRpp = p$	$\varepsilon.NNRR'6R'6.19$
21.	$RRpqNp = p$	$\varepsilon.R'3\varepsilon 6.20$
22.	$NRRRpqNqRRpqNp = NRpq$	$\varepsilon.16.3$
23.	$NRRqRpqNqRRqRpqNp = RRpqNq$	$\varepsilon.16.10$
24.	$NRqRRqNpNp = RRpqNq$	$\varepsilon NR.21.R.15.'.23$
25.	$NRqRNpNq = RRpqNq$	$\varepsilon NR.'.10.24$
26.	$RpRqNp = Np$	$\varepsilon R6'4$
27.	$RRpqNq = q$	$\varepsilon\varepsilon N.26.25.6$
28.	$NRpq = NRqp$	$\varepsilon\varepsilon.22.NR.27.21$
29.	$Rpq = Rqp$	$\varepsilon\varepsilon N.28.6.6$
*30.	$NRpRqr = RRNqpRNrp$	$\varepsilon\varepsilon.29.11.R.29.29$
31.	$RRprRpNq = NRpRqNr$	$\varepsilon RR'6'.11$
32.	$RrRpNq = RRRqppr$	$\varepsilon R'.10.29$
33.	$RRRqppRpq = NRpRqNq$	$\varepsilon.32.31$
34.	$RRpRqpRqp = NRpRqNq$	$\varepsilon R.29.29.33$
35.	$NRpRqNq = p$	$\varepsilon\varepsilon.10.34.27$
*36.	$RpRqNq = Np$	$\varepsilon 6N.35$

Note that

if m is $\alpha = \beta$, $\varepsilon m7$ is $\beta = \alpha$
if m is $N\alpha = N\beta$, $\varepsilon\varepsilon Nm66$ is $\alpha = \beta$.

4. *Second abridgement of Sheffer* (1967). G. Spencer Brown has abridged Sheffer's postulates to the pair

1. $RNpRNqq = p$
2. $RpRqr = NRRNrpRNqp$

with Np for Rpp. One might try abridging this by replacing Nq by p in 1 and shifting the initial N to *LHS* from *RHS*, which effects a shortening when the axiom is written out in full. However, this pair

1. $RRppRpq = p$
2. $RRpRqrRpRqr = RRRrrpRRqqp$

is verified by

R	0	1	2
0	1	1	1
1	1	0	2
2	1	2	2

for which $NNp = p$, $Rpq = Rqp$, $R0p = 1$, $R1p = Np$, but $RpNp = (1,1,2)$, so that for $p/1$, $q/2$, $RpNp \neq RqNq$. However, a modification of 2 gives a pair that works, thus:

1.	$RRppRpq = p$	
2.	$RRpRppRqRrs = RRRssqRRrrq$	
3.	$Rpp = Np$	Df. N
4.	$RNpRpq = p$	$\varepsilon R3'1$
5.	$RNpNp = p$	$\varepsilon R'34$
*6.	$NNp = p$	$\varepsilon 35$
7.	$p = p$	$\varepsilon 66$
8.	$RRRssqRRrrq = RRpNpRqRrs$	$\varepsilon 2RR'3'$
9.	$RRpNpRqRrs = RRNsqRNrq$	$\varepsilon 8RR3'R3'$ (the 3's are not the same)
10.	$RRpNpRqNr = RRNrqRNrq$	$\varepsilon R'R'39$
11.	$NRNNrq = RRpNpRqr$	$\varepsilon 3\varepsilon.10.R'R'6$
12.	$NRrq = RRpNpRqr$	$\varepsilon NR6'.11$
13.	$RRpNpq = Nq$	$\varepsilon R'5\varepsilon.12.N5$
14.	$NRqr = NRrq$	$\varepsilon.13.\varepsilon.12.8$
15.	$Rqr = Rrq$	$\varepsilon\varepsilon N.14.6.6$
*16.	$RqRpNp = Nq$	$\varepsilon.15.13$
17.	$NRqRrs = RRNsqRNrq$	$\varepsilon.13.9$

5. *Third abridgement of Sheffer* (1967).

*1.	$RRppRqp = p$	
2.	$RpRqRpr = RRRrqqp$	
3.	$p = p$	$\varepsilon 11$
4.	$RRRqRqqRqqp = Rpq$	$\varepsilon 2R'1$
5.	$RRRqRqqRqqRRpRppRpp = Rqp$	$\varepsilon\varepsilon 434$
6.	$RRpRppRpp = p$	$\varepsilon\varepsilon R5511$
7.	$Rpq = Rqp$	$\varepsilon R6'4$
8.	$RpRqRpp = Rpp$	$\varepsilon R1'1$
9.	$RRRpqqp = Rpp$	$\varepsilon 28$
10.	$RpRqRpq = Rpp$	$\varepsilon R'7\varepsilon 79$
11.	$RRRqqqp = Rpp$	$\varepsilon 2.10$
12.	$RRRrppRqRrr = RRRrqqRRrpp$	$\varepsilon R'R'92$
13.	$RRRrqqRRrrr = RRqRrrRqRrr$	$\varepsilon 12.11$
14.	$RRRrqqRRrqq = RRqRrrRqRrr$	$\varepsilon\varepsilon 7.11.13$
15.	$RRrqq = RqRrr$	$\varepsilon\varepsilon R14.14.1.1$
16.	$RRqRrrp = RpRqRpr$	$\varepsilon R15.'\varepsilon 2.3$
17.	$RpRqRpr = RpRqRrr$	$\varepsilon 16.7$
18.	$RpRqRrp = RpRqRrr$	$\varepsilon R'R'7.17$
19.	$RRqpRqRrRpp = RRqpRqRrr$	$\varepsilon R'17.18$
20.	$RRqRppRqRrp = RRqRppRqRrr$	$\varepsilon R'R'R'1.19$
21.	$RpRRqpRqp = Rqp$	$\varepsilon 7\varepsilon R'1.1$
22.	$RRqRrpRqRpp = RRqRrpRqRrp$	$\varepsilon 19.R'R'21$

23.	$RRqRppRqRrr = RRqRrpRqRrp$	ε20.ε7.22
24.	$RqRrr = RRrqq$	ε15.3
*25.	$RRqRrpRqRrp = RRRpqqRRrqq$	ε23.R24.24.

(The starred equations are the axioms of Section 3).

Giving this basis as three axioms makes the long one absurdly simple: $RRppRqp = p$, $RpRqRpr = RpRqRqr$, $Rpq = Rqp$.

REFERENCES

[1] Meredith, C. A., and A. N. Prior, "Equational Logic," *Notre Dame Journal of Formal Logic*, vol. IX (1968), pp. 212-226.

Trinity College
Dublin, Ireland